Prof. M. Einsiedler
Prof. P. Biran

Musterlösung Serie 10

1. Consider the real vector space $M_{2\times 2}(\mathbb{R})$.

   (a) Compute the square of
   $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   (b) Find a formula for the entries of the n-th power of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for all $n \in \mathbb{N}$.

   Solution:

   (a) We have
   $$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}.$$

   (b) We show by induction that for all $n \geqslant 1$
   $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$
   For $n = 1$, we do not have anything to prove. Mow let $n \geqslant 1$ and assume that we have proved it for all $k \leqslant n$. We have
   $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix},$$
   where we used the induction hypothesis to obtain the second-to-last equality.

2. Let $\mathbb{R}[X]_n$ be the vectorspace of all polynomials of degree $\leqslant n$ with real coefficients.

   (a) Show that
   $$F : \mathbb{R}[x]_n \to \mathbb{R}[x]_n, \quad p \mapsto p'' + p'$$
   is a linear map, where p' denotes the derivative of p.

   (b) Determine the matrix of F with respect to the basis $(1, x, \dots, x^n)$ of $\mathbb{R}[X]_n$.

   Solution:

   (a) For any $p, q \in \mathbb{R}[X]_n$, for any $\lambda \in \mathbb{R}$, we have
   $$F(p + \lambda q) = (p + \lambda q)'' + (p + \lambda q)' = p'' + \lambda q'' + p' + \lambda q'.$$
   Rearranging, we obtain $F(p + \lambda q) = F(p) + \lambda F(q)$. We conclude that F is linear.

(b) We compute

$$F(x^i) = \begin{cases} 0, & \text{if } i = 0 \\ 1, & \text{if } i = 1 \\ i(i-1)x^{i-2} + ix^{i-1}, & \text{if } i \geqslant 2 \end{cases}$$

Denoting $\mathcal{B} = \{1, x, x^2, \cdots, x^n\}$ the standard basis of $\mathbb{R}[x]_n$, we write it as

$$_{\mathcal{B}}[F]_{\mathcal{B}} = ((j-1)(j-2)\delta_{i,j-2} + (j-1)\delta_{i,j-1})_{1 \leqslant i,j \leqslant n+1}.$$

In the above notation everything is offset by -1 since the power of the polynomials in the basis start with 0 but the indexing of the matrix starts with 1. More explicitly, it will look as follows:

$$_{\mathcal{B}}[F]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 6 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

3. Let K be a field.

(a) Consider the matrices

$$A = \begin{pmatrix} A_1 & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & A_2 \end{pmatrix} \in M_{n\times n}(K)$$

with $A_1 \in M_{k\times k}(K)$ and $A_2 \in M_{(n-k)\times(n-k)}(K)$ for some $k \geqslant 1$, and

$$B = \begin{pmatrix} B_1 & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & B_2 \end{pmatrix} \in M_{n\times n}(K)$$

with $B_1 \in M_{k\times k}(K)$ and $B_2 \in M_{(n-k)\times(n-k)}(K)$. Show that

$$A \cdot B = \begin{pmatrix} A_1 \cdot B_1 & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & A_2 \cdot B_2 \end{pmatrix}$$

(b) Let A be defined as above and assume that A_1, respectively A_2, is invertible as an element of $M_{k\times k}(K)$, respectively $M_{(n-k)\times(n-k)}(K)$. Show that A is invertible.

(c) Consider the space U of upper triangular matrices in $M_{n\times n}(K)$. Show that the product of 2 elements of U is in U.

Solution:

(a) Let us denote $C := A \cdot B$. We will write $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$. We recall the product formula:

$$\forall\, 1 \leqslant i, j \leqslant n : \quad c_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}.$$

Let us divide the matrices into the quadrants

$$\begin{matrix} 1 \leqslant i, j \leqslant k, & 1 \leqslant i \leqslant k \wedge k < j \leqslant n, \\ k < i \leqslant n \wedge 1 \leqslant j \leqslant k, & k < i, j \leqslant n. \end{matrix}$$

We treat the cases separately.
If $1 \leqslant i, j \leqslant k$, we have

$$c_{ij} = \sum_{\ell=1}^{k} a_{i\ell} b_{\ell j} + \sum_{\ell=k+1}^{n} 0 \cdot 0 = \sum_{\ell=1}^{k} a_{i\ell} b_{\ell j}.$$

The last sum is the formula to compute an entry of the product $A_1 \cdot B_1$ since i, j are arbitrary in $\{1, \ldots, k\}$.
We proceed similarly for the 3 other quadrants and obtain the formula.

(b) Denote A_i^{-1} the inverse of A_i, $i = 1, 2$ and define

$$B := \begin{pmatrix} A_1^{-1} & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & A_2^{-1} \end{pmatrix}.$$

By a),

$$A \cdot B = \begin{pmatrix} A_1 \cdot A_1^{-1} & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & A_2 \cdot A_2^{-1} \end{pmatrix} = \begin{pmatrix} I_k & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & I_{n-k} \end{pmatrix} = I_n,$$

where I_ℓ, $1 \leqslant \ell \leqslant n$, denotes the identity matrix of size $\ell \times \ell$. Hence A is invertible with inverse B in $M_{n \times n}(K)$.

(c) Let $A, B \in M_{n \times n}(K)$ be upper triangular matrices and denote their product C. Let $1 \leqslant i, j \leqslant n$, we have

$$c_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}.$$

Since $a_{i\ell} = 0$ for $\ell < i$ and $b_{\ell j} = 0$ for $\ell > j$, we have

$$c_{ij} = \sum_{\substack{1 \leqslant \ell \leqslant n \\ i \leqslant \ell \leqslant j}} a_{i\ell} b_{\ell j}, \quad \text{if } i \leqslant j$$

and $c_{ij} = 0$ otherwise. Hence C is upper triangular.

4. Let V and W be finite-dimensinal vector spaces over a field K.

   (a) Suppose that $2 \leqslant \dim V \leqslant \dim W$. Show that
   $$\mathcal{S} := \{T \in \mathrm{Hom}(V, W) \mid T \text{ is not injective}\}$$
   is not a subspace of $\mathrm{Hom}(V, W)$.

   (b) Suppose that $\dim V \geqslant \dim W \geqslant 2$. Show that
   $$\mathcal{T} := \{T \in \mathrm{Hom}(V, W) \mid T \text{ is not surjective}\}$$
   is not a subspace of $\mathrm{Hom}(V, W)$.

Solution:

(a) We will show that there exist two elements T and T' of $\mathcal{S}$ such that $T+T' \notin \mathcal{S}$. Let $\{v_1, v_2, \ldots, v_r\} \subset V$ be a basis of V and let $\{w_1, w_2, \ldots, w_s\} \subset W$ be a basis of W. By assumption, we have $2 \leqslant r \leqslant s$. We consider an element T of $\mathcal{S}$ defined as follows: we set $T(v_1) = w_1$ and $T(v_i) = 0$ for $2 \leqslant i \leqslant r$ and extend it linearly to obtain a linear map on the whole of V. Then $\dim(\mathrm{Kern}(T)) = r-1$ and $\mathrm{rank}(T) = 1$. So, T isn't injective and T isn't the 0 map, i.e. $T \in \mathcal{S} \setminus \{0\}$. We now define T'. We want to build it so that
$$\forall v \in V : 0 = (T+T')(v) = T(v) + T'(v) \Leftrightarrow v = 0.$$
We set $T'(v_1) = 0$, $T'(v_i) = w_i$ for all $2 \leqslant i \leqslant r$ and extend it linearly to obtain an element of $\mathrm{Hom}(V, W)$. Assume now that $v = \sum_{i=1}^{r} a_i v_i \in \mathrm{Kern}(T+T')$. Then
$$0 = (T+T')\left(\sum_{i=1}^{r} a_i v_i\right) = \sum_{i=1}^{r} a_i(T(v_i) + T'(v_i)) = \sum_{i=1}^{r} a_i w_i.$$
Since $\{w_1, w_2, \ldots w_r\}$ is a linearly independent subset of W, this implies $\forall 1 \leqslant i \leqslant r : a_i = 0$ and therefore, $v = 0$. This shows that $T + T' \notin \mathcal{S}$.

(b) Let $\{v_1, v_2, \ldots, v_r\}$ be a basis of V and let $\{w_1, w_2, \ldots, w_s\}$ be a basis of W. This time, we have $2 \leqslant s \leqslant r$. We define $T \in \mathcal{T}$ exactly as above. It is not surjective since $\mathrm{Bild}(T) \subseteq \mathrm{Sp}(w_1) \not\ni w_2$. We define T' on the basis as follows: we let $T'(v_1) = 0$; for $2 \leqslant i \leqslant s$, we let $T'(v_i) = w_s$; for $s < i \leqslant r$ we let $T'(v_i) = w_s$. We extend T' linearly to an element of $\mathrm{Hom}(V, W)$. Since $\mathrm{Bild}(T') \subseteq \mathrm{Sp}(w_2, \cdots, w_s) \not\ni w_1$, $T' \in \mathcal{T}$.
We check that $T + T'$ is surjective. Let $w = \sum_{i=1}^{s} a_i w_i \in W$. Then
$$\begin{aligned} w = a_1 T(v_1) + \sum_{i=2}^{s} a_i T'(v_i) &= \sum_{i=1}^{s} a_i(T(v_i) + T'(v_i)) \\ &= \sum_{i=1}^{s} (T+T')(a_i v_i) \in \mathrm{Bild}(T+T'). \end{aligned}$$
Hence $T + T' \notin \mathcal{T}$.

5. Consider the linear maps $\mathbb{R}^4 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3$ given by

$$f : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 2x_2 + x_3 \\ x_1 - x_4 \end{pmatrix} \quad \text{and} \quad g : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 \end{pmatrix}.$$

Let

$$\mathcal{A} := \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix} \right),$$

and let $\mathcal{B}$ be the standard basis of $\mathbb{R}^2$ and let

$$\mathcal{C} := \left(\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right).$$

(a) Show that $\mathcal{A}$ is a basis of $\mathbb{R}^4$ and that $\mathcal{C}$ is a basis of $\mathbb{R}^3$.

(b) Determine $g \circ f$ and the matrices

  (i) of f with respect to the bases $\mathcal{A}, \mathcal{B}$.

  (ii) of g with respect to the bases $\mathcal{B}, \mathcal{C}$.

  (iii) of $g \circ f$ with respect to the bases $\mathcal{A}, \mathcal{C}$.

Lösung:

(a) Gaussion elimination yields that the matrices

$$A := \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 4 & 1 & 2 \end{pmatrix}$$

are invertible. This proves that $\mathcal{A}$ is a basis of $\mathbb{R}^4$ and $\mathcal{C}$ is a basis of $\mathbb{R}^3$.

(b) The maps f and g can be represented by left multiplication with matrices, i.e. $f = T_U$ and $g = T_V$ for

$$U := \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 0 \end{pmatrix}.$$

Thus we have $g \circ f = T_V \circ T_U = T_{VU}$ for the matrix

$$VU = \begin{pmatrix} 2 & 2 & 1 & -1 \\ 0 & 2 & 1 & 1 \\ 3 & 6 & 3 & 0 \end{pmatrix}.$$

The matrices U, V, VU represent $f, g, g \circ f$ with respect to the standard bases $\mathcal{B}_n$ of $\mathbb{R}^n$, which means that

$$U = [f]_{\mathcal{B}_4}^{\mathcal{B}_2}, \quad V = [g]_{\mathcal{B}_2}^{\mathcal{B}_3} \quad \text{and} \quad VU = [g \circ f]_{\mathcal{B}_4}^{\mathcal{B}_3}.$$

The matrix A is the base change matrix $A = {}_{\mathcal{B}_4}[\mathrm{id}_{\mathbb{R}^4}]_{\mathcal{A}}$ and C is the base change matrix $C = {}_{\mathcal{B}_3}[\mathrm{id}_{\mathbb{R}^3}]_{\mathcal{C}}$.

(i) The matrix representing f w.r.t. $\mathcal{A}$ and $\mathcal{B}$ is

$$\begin{aligned} [f]_{\mathcal{A}}^{\mathcal{B}} &= [f]_{\mathcal{B}_4}^{\mathcal{B}} \cdot [\mathrm{id}_{\mathbb{R}^4}]_{\mathcal{A}}^{\mathcal{B}_4} \\ &= U \cdot A \\ &= \begin{pmatrix} 2 & 11 & 4 & 5 \\ 1 & -1 & 0 & 2 \end{pmatrix}. \end{aligned}$$

(ii) We have

$$\begin{aligned} [g]_{\mathcal{B}}^{\mathcal{C}} &= [\mathrm{id}_{\mathbb{R}^3}]_{\mathcal{B}_3}^{\mathcal{C}} \cdot [g]_{\mathcal{B}}^{\mathcal{B}_3} \\ &= \left([\mathrm{id}_{\mathbb{R}^3}^{-1}]_{\mathcal{C}}^{\mathcal{B}_3}\right)^{-1} \cdot [g]_{\mathcal{B}}^{\mathcal{B}_3} \\ &= \begin{pmatrix} -1 & -1 \\ -1 & 0 \\ 4 & 2 \end{pmatrix}. \end{aligned}$$

(iii) The matrix representing $g \circ f$ w.r.t $\mathcal{A}$ and $\mathcal{C}$ is

$$\begin{aligned} [g \circ f]_{\mathcal{A}}^{\mathcal{C}} &= [g]_{\mathcal{B}}^{\mathcal{C}} \cdot [f]_{\mathcal{A}}^{\mathcal{B}} \\ &= \begin{pmatrix} -3 & -10 & -4 & -7 \\ -2 & -11 & -4 & -5 \\ 10 & 42 & 16 & 24 \end{pmatrix}. \end{aligned}$$

6. Let V be a vector space over a field K. Suppose that T_1 and T_2 are two linear maps from V to K that have the same kernel. Show that there exists a constant $c \in K$ such that $T_1 = cT_2$.

Hint: Use Serie 9 exercise 1.

Solution: If T_1 or T_2 is the 0-map, we are done. Assume that T_2 isn't the 0-map, i.e. that $\ker(T_2) \subsetneq V$. Let $u \in V \setminus \ker(T_2)$. Then, $v_0 := \frac{u}{T_2(u)}$ is such that $T_2(v_0) = 1 \in K$. Let $v \in V$ and consider

$$w = v - T_2(v)v_0 \in V.$$

It is an element of $\ker(T_2)$ since

$$T_2(w) = T_2(v) - T_2(v)T_2(v_0) = T_2(v) - T_2(v) = 0.$$

We assumed that $\ker(T_1) = \ker(T_2)$, therefore

$$0 = T_1(v) = T_1(v) - T_2(v)T_1(v_0) \Leftrightarrow T_1(v) = T_1(v_0)T_2(v).$$

Since $v \in V$ was arbitrary, we set $c := T_1(v_0)$ and conclude that

$$T_1 = cT_2.$$