D-MATH
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Musterlösung Serie 11

1. Suppose that V, W are finite-dimensional vector spaces over a field K. Let $T \in \mathrm{Hom}(V, W)$. Prove that $\mathrm{rank}(T) = 1$ if and only if there exists a basis $\mathcal{B}$ of V and a basis $\mathcal{C}$ of W such that with respect to these bases, all entries of $[T]^{\mathcal{B}}_{\mathcal{C}}$ equal 1.

   Solution: We write m the dimension of V and n the dimension of W and denote $\mathcal{B}_0$, respectively $\mathcal{C}_0$, an arbitrary basis of V. respectively W. Since T is a homomorphism of rank 1, the matrix $[T]^{\mathcal{B}_0}_{\mathcal{C}_0}$ is of rank 1. It is therefore equivalent to any matrix of rank 1 in $M_{n\times m}(K)$, namely there exist matrices $P \in \mathrm{GL}_n(K)$ and $Q \in \mathrm{GL}_m(K)$ such that $P[T]^{\mathcal{B}_0}_{\mathcal{C}_0}Q = A$, where $A \in M_{n\times n}(K)$ is the matrix whose entries are all equal to 1. Since $Q \in \mathrm{GL}_m(K)$, it can be seen as the base change matrix from $\mathcal{B}$ to $\mathcal{B}_0$, where $\mathcal{B}$ is defined such that the diagram

   $$\begin{array}{ccc} V & \xrightarrow{\mathrm{Id}_V} & V \\ \downarrow{\scriptstyle \Phi_{\mathcal{B}}} & & \downarrow{\scriptstyle \Phi_{\mathcal{B}_0}} \\ K^m & \xrightarrow{P} & K^m \end{array}$$

   commutes. Similarly, C defines a base-change matrix from $\mathcal{C}_0$ to some basis $\mathcal{C}$. Hence,

   $$A = P[T]^{\mathcal{B}_0}_{\mathcal{C}_0}Q = [\mathrm{Id}_W]^{\mathcal{C}\prime}_{\mathcal{C}}[T]^{\mathcal{B}_0}_{\mathcal{C}_0}[\mathrm{Id}_V]^{\mathcal{B}}_{\mathcal{B}_0} = [T]^{\mathcal{B}}_{\mathcal{C}}.$$

2. Suppose V is finite-dimensional and $S, T, U \in \mathrm{Hom}(V, V)$ with $STU = \mathrm{Id}_V$. Show that T is invertible and that $T^{-1} = US$.

   Solution: Observe that $STU = \mathrm{Id}_V$ implies that STU is an automorphism of V. Since $\{0\} = \mathrm{Ker}(\mathrm{Id}_V) = \mathrm{Ker}(STU) \supseteq \mathrm{Ker}(U)$, we must have that U is injective. Since V is finite-dimensional, this implies that U is bijective and hence invertible. So,

   $$ST = U^{-1}.$$

   Also since $V = \mathrm{Im}(STU) \subseteq \mathrm{Im}(S)$, we deduce that S is surjective. Since $S : V \to V$ and V is finite-dimensional, this implies that S is injective. Hence S is invertible and

   $$T = S^{-1}U^{-1}.$$

   This shows both that T is invertible and that

   $$T^{-1} = US.$$

3. Let V be a finite-dimensional vector spaces over a field K. Suppose that $T \in \mathrm{Hom}(V, V)$, and that $\mathcal{A} = \{u_1, \dots, u_n\}$ and $\mathcal{B} = \{v_1, \dots, v_n\}$ are bases of V. Prove that the following are equivalent:

(a) T is invertible.
(b) The columns of $[T]_{\mathcal{B}}^{\mathcal{A}}$ are linearly independent in K_{col}^{n};
(c) The columns of $[T]_{\mathcal{B}}^{\mathcal{A}}$ span K_{col}^{n};
(d) The rows of $[T]_{\mathcal{B}}^{\mathcal{A}}$ are linearly independent in K_{row}^{n};
(e) The rows of $[T]_{\mathcal{B}}^{\mathcal{A}}$ span K_{row}^{n}.

Solution: See the proof of proposition 3.6.8 (p.128) in Menny Aka's lecture notes, available on the course website.

4. Let V be a finite-dimensional vector space. Prove or disprove:

(a) Let $V' \subset V$ be a linear subspace. Every automorphism $f : V' \to V'$ can be extended to an automorphism $\bar{f} : V \to V$.
(b) For every Endomorphism $f : V \to V$ its image $\mathrm{Im}(f)$ is a linear complement $\mathrm{Ker}(f)$ in V.
(c) There does not exist any linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\operatorname{rank}(T) = \dim \operatorname{Ker}(T).$$

Lösung:

(a) This assertion is true. Choose a complement of V', i.e., a subvectorspace V'' of V with $V = V' + V''$ and $V' \cap V'' = \{0\}$. Then the map

$$V' \times V'' \to V, (v', v'') \mapsto v' + v''$$

is bijective. Define a map $\bar{f} : V \to V$ by $\bar{f}(v' + v'') := f(v') + v''$ for all $v' \in V'$ and $v'' \in V''$. As f is linear, a direct computation shows that $\bar{f}$ is linear as well. We claim that $\bar{f}$ is bijective. To see this, consider first $v' \in V'$ and $v'' \in V''$ with $f(v') + v'' = 0$. Then we have $f(v') = -v'' \in V' \cap V'' = \{0\}$ and so $f(v') = v'' = 0$. As f is injective, we also get $v' = 0$. Thus, we have $\mathrm{Ker}(\bar{f}) = \{0\}$ and hence $\bar{f}$ is injektive.
Now let $v \in V$ be an arbitrary element. Write $v = v' + v''$ with $v' \in V'$ and $v'' \in V''$. Then, we have $v' = f(f^{-1}(v'))$ and thus $v = f(f^{-1}(v')) + v'' = \bar{f}(f^{-1}(v') + v'')$. Therefore, the map $\bar{f}$ is surjektive. Together we get that $\bar{f}$ is bijektive and thus an isomorphism $V \to V$, and hence an Automorphismus of V.

(b) The assertion is false. Consider for example the endomorphism $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by left multiplication with

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then, we have $\mathrm{Ker}(L_A) = \langle (1,0)^T \rangle = \mathrm{Bild}(L_A)$. The subspaces neither have intersection $\{0\}$, nor do they generate $\mathbb{R}^2$, and thus they cannot form $V = \mathrm{Ker}(f) \oplus \mathrm{Bild}(f)$.

(c) Consider an endomorphism $T : \mathbb{R}^5 \to \mathbb{R}^5$. By the rank theorem,

$$\operatorname{rank}(T) + \dim(\operatorname{Ker}(T)) = \dim(\mathbb{R}^5) = 5.$$

If we were to have $\operatorname{rank}(T) = \dim(\operatorname{Ker}(T))$, we would have

$$5 = 2\operatorname{rank}(T) \in 2\mathbb{Z},$$

which cannot hold since 5 is not even.

5. Consider the space $M_{2\times 2}(K)$ of 2-by-2 matrices over a field K.

   (a) Show that if $A \in M_{2\times 2}(K)$ satisfies $A^2 \neq 0$, then $A^k \neq 0$ for all $k \geqslant 3$.

   (b) Find a field K and a matrix $A \in M_{2\times 2}(K) \setminus \{0\}$ such that $\exists n \in \mathbb{N} : A^n = 0$.

   Solution:

   (a) Let $A \in M_{2\times 2}(K)$ and denote $T = T_A$ be the linear map associated to A. If $T(K^2) = K^2$, then T is an isomorphism and $T^k \neq 0$ for all $k \geqslant 1$. Assume now that $T(K^2) = L$ is a one-dimensional subspace of K^2. Then $T^2(K^2) = T(L) \subseteq L$. However, since $T^2 \neq 0$, $T^2(K^2) = T(L) = L$.
   Let $n \geqslant 2$. We now assume that $T^m(K^2) = L$ for $2 \leqslant m \leqslant n$ and we show that $T^{n+1}(K^2) = L$. Indeed,

   $$T^{n+1}(K^2) = T(T^n(K^2)) = T(L) = L.$$

   This shows that $T^k \neq 0$ for all $k \geqslant 1$.

   (b) Take for example

   $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{C}).$$

6. Determine the ranks of the following rational $n \times n$-matrixes, those being elements of $M_{n\times n}(\mathbb{Q})$, depending on the positive integer n.

   (a) $(kl)_{k,l=1,\ldots,n}$;

   (b) $\big((-1)^{k+l}(k+l-1)\big)_{k,l=1,\ldots,n}$;

   (c) $\left(\dfrac{(k+l)!}{k!l!}\right)_{k,l=0,\ldots,n-1}$.

   Hint: Note that the last matrix is indexed from 0 to $n-1$. You may use induction to find the desired formula.

   Solution:

(a) Define $B := (b_{kl})_{k,l=1,\ldots,n} := (kl)_{k,l=1,\ldots,n}$. Let $B' = (b'_{kl})_{k,l=1,\ldots,n}$ be the matrix arising from B by substracting for every $k = 2, \ldots, n$ the k-multiple of the first row from the k-st row. Then, we have

$$b'_{kl} = \begin{cases} b_{kl} = l & \text{falls } k = 1 \\ b_{kl} - kb_{1l} = kl - kl = 0 & \text{falls } k > 1. \end{cases}$$

Thus, the matrix B' has exactly one non-vanishing row and thus has rank 1. As B' arose through elementary row operations from B, and hence by left multiplication with an invertible matrix, the rank of B' is the same as the rank of B. Therefore, we get $\mathrm{Rang}(B) = 1$.

Aliter: Let $u := (1, \ldots, n)$ the $1 \times n$ matrix with entry i on the position $(1, i)$. Then, we have $B = u^T \cdot u$. As the rank of (u) is $\leqslant 1$, exercise 3 (c) yields $\mathrm{Rang}(B) \leqslant 1$. From $B \neq 0$, we also get $\mathrm{Rang}(B) \geqslant 1$ and thus $\mathrm{Rang}(B) = 1$.

(b) Let $B := (b_{kl})_{k,l=1,\ldots,n}$ with $b_{kl} := (-1)^{k+l}(k+l-1)$. For $n = 1$, the matrix$B = (1) \neq 0$ has rank 1. It remains to treat the case $n \geqslant 2$. For all $k = 1, \ldots, n-2$ and $l = 1, \ldots, n$, we get

$$b_{kl} + 2b_{k+1,l} + b_{k+2,l} = 0,$$

and thus the k-th row of B is a linear combination of the $(k+1)$-th and $(k+2)$-th row. Therefore, the matrix B can by elementary row operations be transformed into a matrix, in which only the last two rows are non-vanishing and these are identical to the last tow rows of B. These two are linearly independent, which we get from a direct computation. Together, we have

$$\mathrm{rank}(B) = \begin{cases} 1, & \text{falls } n = 1 \\ 2, & \text{falls } n \geqslant 2 \end{cases}$$

(c) Let $C_n := (c_{kl})_{k,l=0,\ldots,n-1}$ be the matrix with $c_{kl} := \frac{(k+\ell)!}{k!\ell!} = \begin{pmatrix} k+l \\ l \end{pmatrix}$.

Claim: $\mathrm{rank}\,(C_n) = n$.

Proof: We induct over n. For $n = 1$, the assertion is true, as $C_1 = (1) \neq 0$. Assume, that we know the claim for $n \geqslant 1$. Let $C' = (c'_{kl})_{k,l=0,\ldots,n}$ be the matrix, which arises from C_{n+1}, when beginning with the last row, from each row the previous row is substracted. In other words:

$$c'_{kl} := \begin{cases} c_{kl} - c_{k-1,l} & \text{if } k = 1, \ldots, n \\ c_{0l} & \text{if } k = 0. \end{cases}$$

Let moreover $C'' = (c''_{kl})_{k,l=0,\ldots,n}$ be the matrix, which arises from C', when beginning with the last column, from each column the previous column is substracted. In other words:

$$c''_{kl} := \begin{cases} c'_{kl} - c'_{k,l-1} & l = 1, \dots, n \\ c'_{k0} & l = 0. \end{cases}$$

For every $1 \leqslant k, l \leqslant n$, we have

$$\begin{aligned} c''_{kl} &= c'_{kl} - c'_{k,l-1} \\ &= (c_{kl} - c_{k-1,l}) - (c_{k,l-1} - c_{k-1,l-1}) \\ &= \frac{(k+\ell)!}{k!\ell!} - \frac{(k+\ell-1)!}{(k-1)!\ell!} - \frac{(k+\ell-1)!}{k!(\ell-1)!} + \frac{(k+\ell-2)!}{(k-1)!(\ell-1)!} \\ &= \frac{(k+\ell)!}{k!\ell!}\left(1 - \frac{k}{k+1} - \frac{l}{k+l}\right) + \frac{(k+\ell-2)!}{(k-1)!(\ell-1)!} \\ &= \frac{(k+\ell-2)!}{(k-1)!(\ell-1)!}. \end{aligned}$$

Therefore, we get

$$C'' = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & C_n & \\ 0 & & & \end{pmatrix}$$

and finally

$$\text{Rang } C_{n+1} = \text{Rang}\,(C'') = 1 + \text{ Rang } C_n = n + 1.$$