D-MATH | Lineare Algebra I | HS 2022

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Musterlösung Serie 12

1. For each of the following matrices, determine whether or not it is invertible, and if it is compute its inverse.

   (a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 2 & -3 & 1 \\ -1 & 3 & -3 & -2 \\ 2 & 0 & 1 & 5 \\ 3 & 1 & -2 & 6 \end{pmatrix}$

   Lösung:

   (a) This matrix is of rank 1 since the first row is equal to the second and to the third. Therefore it cannot be transformed into the identity using elementary row-operations and it isn't invertible.

   (b) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 4/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & 4/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & 4/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & 4/5 \end{pmatrix}$

   (d) $\begin{pmatrix} 35/11 & -16/11 & 13/11 & -2 \\ 1 & 0 & 1 & -1 \\ 10/11 & -3/11 & 10/11 & -1 \\ -16/11 & 7/11 & -5/11 & 1 \end{pmatrix}.$

2. Show that a square matrix A is invertible if and only if its transpose A^T is invertible. Moreover, show that in that case we have $(A^T)^{-1} = (A^{-1})^T$.

   Lösung: We already know for all quadratic matrices A and B of the same size, the equation $(A \cdot B)^T = B^T \cdot A^T$ is satisfied.

   First, assume that A is invertible with inverse A^{-1}. Then:

$$A^T \cdot (A^{-1})^T = (A^{-1} \cdot A)^T = I^T = I,$$

$$(A^{-1})^T \cdot A^T = (A \cdot A^{-1})^T = I^T = I.$$

   Therefore, the matrix A^T is invertible with inverse $(A^{-1})^T$ and we have $(A^T)^{-1} = (A^{-1})^T$.

   Second, assume A^T is invertible. We proved in the lecture $(A^T)^T = A$. Going through the first case with A^T instead of A yields that $(A^T)^T = A$ is invertible.

3. (a) Let V, W be two n-dimensional vector spaces over a field K. Let $S \in \mathrm{Hom}(V, W)$ and $T \in \mathrm{Hom}(W, V)$ such that $T \circ S = \mathrm{Id}_V$. Show that S is invertible and T is the inverse of S.

   (b) Let $A, B \in M_{n\times n}(K)$ such that $A \cdot B = I_n$. Show that A is invertible with inverse B.

   Lösung:

   (a) Since S admits a left-sided inverse, it is injective. Since W is finite-dimensional, this implies that S is surjective. Since T admits a right-sided inverse, it is surjective. So, it is injective as V is finite-dimensional. This shows that both S and T are isomorphisms, and therefore that they are invertible. Note that for any $w \in W$, there exists a unique $v \in V$ such that $S(v) = w$. Hence

$$S \circ T(w) = S \circ T(S(v)) = S(v) = w.$$

   So,

$$T \circ S = \mathrm{Id}_V \quad \text{and} \quad S \circ T = \mathrm{Id}_W,$$

   which implies that T is the inverse of S by definition.

   (b) A, respectively B, defines a linear map $T_A : K^n_{\text{col}} \to K^n_{\text{col}}$, $v \mapsto A \cdot v$, respectively $T_B : K^n_{\text{col}} \to K^n_{\text{col}}$, $v \mapsto B \cdot v$. Since $A \cdot B = I_n$, for any $w \in K^n_{\text{col}}$ we have

$$T_A \circ T_B(w) = T_A(B \cdot w) = A \cdot (B \cdot w) = (A \cdot B) \cdot w = w.$$

   So, $T_A \circ T_B = \mathrm{Id}_{K^n_{\text{col}}}$ and by (a), we obtain that T_A and T_B are invertible and are the inverse of each other. Equivalently

$$B \cdot A = A \cdot B = I_n,$$

   so A and B are invertible with $B = A^{-1}$.

4. Consider $n \times n$-matrices A and B over K.

   (a) Show: If A or B is invertible, then AB and BA are similar.

   (b) Does this also hold true without the condition in (a)?

   Lösung: (a) If A is invertible, then $AB = A(BA)A^{-1}$ is similar to BA. Likewise, if B is invertible, then $BA = B(AB)B^{-1}$ is similar to AB.

   (b) For $A := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we have $AB = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $BA = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. As the zeromatrix is only similar to itself, this is a counterexample.

5. Determine with the help of Gaussian elimination, for which values of α the following matrix over $\mathbb{Q}$ is invertible:

$$\begin{pmatrix} 1 & 3 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ 3 & -1 & -2 & -2\alpha \\ -6 & 2 & 1 & \alpha^2 \end{pmatrix}.$$

   Lösung: Gaussian elimination yields the matrix

$$\begin{pmatrix} 1 & 3 & -4 & 2 \\ 0 & -5 & 6 & -3 \\ 0 & 0 & -2 & -2\alpha \\ 0 & 0 & 0 & \alpha^2 - \alpha \end{pmatrix},$$

   which is invertible if and only if the original matrix is invertible. As a triangular matrix is invertible if and only if it has non-zero diagonal entries, it follows that the original mamtrix is invertible if and only if

$$\alpha^2 - \alpha = \alpha(\alpha - 1) \neq 0.$$

   This is equivalent to $\alpha \notin \{0, 1\}$.

6. Prove the following:

   Theorem. *(Bruhat-Decomposition)* For every invertible matrix A there exists a permutation matrix P, i.e. a matrix which has exactly one non-zero, which equals 1, in every column and every row, and invertible upper triangular matrices B and B', such that

$$A = BPB'.$$

   Hint: Choose an invertible upper triangular matrix U, such that the sum of the number of leading zeros in all rows of UA is maximal. Then find a permutation matrix Q, such that QUA is an invertible upper triangular matrix.

Multiplying a matrix A on the left by a permutation matrix P permutes the rows of A. For example, if P has a 1 at index (i, j), the j–th line of A will become the i-th line of PA.

Lösung: For every invertible upper triangular matrix U we consider the sum of the nubmer of leading zeros of all rows of UA. This sum is bounded by the number of entries of A, and hence we can choose an invertible upper triangular matrix U which minimises this sum.

Claim. The numbers of leading zeros of the rows of UA are pairwise different.

Proof. Assume that the claim is false for the i-th and j-th row with $i < j$. Then the first non-zero entry of these rows is in the same cloumn. We substract a suitable multiple of the j-th row from the i-th row. The i-th row of the resulting matrix has at least one leading zero more. This row operation corresponds to left multiplicationwith an invertible upper triangular matrix of the Form $T = I_m + \lambda E_{ij}$. As the other rows are unchanged, the sum of the leading zeros of TUA is greater than the one of UA. As TU also is an invertible upper triangular matrix, this is a contradiction to the maximised choice of U. □

The claim yields that we can transform UA into an upper triangular matrix B' by left multiplication with a permutation matrix Q. In other words, we have $QUA = B'$.As Q, U and A are invertible, so is B'. The inverse of a permutation matrix is again a permutation matrix and the inverse of an upper triangular matrix is again an upper triangular matrix. Hence with $P := Q^{-1}$ and $B := U^{-1}$ we get the equation $A = BPB'$ of the desired form.

Single Choice. In each exercise, exactly one answer is correct.

1. Which assertion is not always satisfied?

   (a) The base change matrix is the representation matrix of the identity map with respect to the according bases.

   (b) Every finite dimensional vector space is isomorpic to K^n for some $n \geqslant 0$.

   ✓ (c) The rank of a linear map $f : K^n \to K^m$ is at least $\min\{n, m\}$.

   (d) The representation matrix of an isomorphism is invertible.

   Erklärung: The rank of a linear map $f\colon K^n \to K^m$ is at *most* $\min\{n, m\}$; but can for example also be 0, thus (c)is false.

2. Consider $\mathbb{C}$ as real two-dimensional vectorspace with the ordered basis $\mathcal{B} := (1, i)$. The matrix $[\dots]^{\mathcal{B}}_{\mathcal{B}} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the representation matrix with respect to $\mathcal{B}$ of the linear map $\mathbb{C} \to \mathbb{C}$:

   (a) Complex conjugation $z \mapsto \bar{z}$

   (b) $z \mapsto \mathrm{Re}(z)$

   (c) $z \mapsto \mathrm{Im}(z)$

   ✓ (d) $z \mapsto iz$

   Erklärung: In the Basis $\mathcal{B}$ an element $a + ib \in \mathbb{C}$ is written as the vecor $\binom{a}{b}$. Under the matrix $[\dots]^{\mathcal{B}}_{\mathcal{B}}$, it is mapped to $\binom{-b}{a}$, which corresponds to the elemtn $-b + ia = i(a + ib)$.

3. The rank of $\begin{pmatrix} 1 & 3 \\ -3 & -9 \end{pmatrix}$ over $\mathbb{Q}$ is

   (a) 0

   ✓(b) 1

   (c) 2

   (d) 3

   Erklärung: The columns of the matrix are non-zero, but linearly dependent. The second column is -3 times the second one. Thus, the rank is 1.

4. For every $n \times m$-Matrix A and every invertible $n \times n$-matrix B, we have

   (a) $\operatorname{rank}(BA) = \operatorname{rank}(B) \cdot \operatorname{rank}(A)$

   (b) $\operatorname{rank}(BA) = \operatorname{rank}(B) + \operatorname{rank}(A)$

   ✓(c) $\operatorname{rank}(BA) = \operatorname{rank}(A)$

   (d) $\operatorname{rank}(BA) = \operatorname{rank}(B)$

   Erklärung: According to Lemma 3.4.2 of the lecture, the rank of a linear map does not change by left composition with an isomorphism. In the language of matrices this translates to (c).

Multiple Choice Fragen.

1. Consider the following ordered bases of $\mathbb{R}[x]_2$:

$$\mathcal{B} = (1, x, x^2), \quad \mathcal{C} = (x^2, (x+1)^2, (x+2)^2)$$

Determine the base change matrix $Q := [\mathrm{Id}_{\mathbb{R}[x]_2}]^{\mathcal{B}}_{\mathcal{C}}$.

(a) $Q = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}$

(b) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{pmatrix}$

(c) $Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 4 & 1 \end{pmatrix}$

(d) $Q = \frac{1}{4}\begin{pmatrix} 2 & -4 & 4 \\ -1 & 4 & -3 \\ 4 & 0 & 0 \end{pmatrix}$

✓ (e) $Q = \frac{1}{4}\begin{pmatrix} 2 & -3 & 4 \\ -4 & 4 & 0 \\ 2 & -1 & 0 \end{pmatrix}$

(f) $Q = \frac{1}{4}\begin{pmatrix} 2 & -1 & 4 \\ -4 & 4 & 0 \\ 4 & -3 & 0 \end{pmatrix}$

Erklärung: One solves the system of equations

$$p = ax^2 + b(x+1)^2 + c(x+2)^2 = (a+b+c)x^2 + (2b+4c)x + (b+4c)$$

for $p = 1, x, x^2$ using comparison of coefficients. For $p = 1$, we get

$$\begin{aligned} b + 4c &= 1 \\ 2b + 4c &= 0 \\ a + b + c &= 0 \end{aligned}$$

The second equation yields $c = -\frac{1}{2}b$ and substitution into the first one $b = -1$. Hence we get $c = \frac{1}{2}$. Substitution into the third equation implies and thus

$$[1]_{\mathcal{C}} = \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$$

This already excludes all but one answers. Of course one could now also compute $[x]_{\mathcal{C}}$, and $[x^2]_{\mathcal{C}}$.

2. Let $A \in M_{n\times n}(K)$ and $P \in \mathrm{GL}_n(K)$, denote

$$r := \operatorname{rank}(A - I_n) \quad \text{and} \quad s := \operatorname{rank}(PAP^{-1} - I_n).$$

Which of the following statements hold?

✓ (a) $r = s$;
Erklärung: Note that

$$P(A - I_n)P^{-1} = PAP^{-1} - PI_nP^{-1} = PAP^{-1} - I_n.$$

Hence $PAP^{-1} - I_n$ and $A - I_n$ are equivalent matrices. As seen in the lectures, this implies $r = s$.

(b) $r \neq s$;

(c) $r > s$;

(d) $r < s$;

✓ (e) if $r < n$, $\exists v \in V \setminus \{0\}$ such that $Av = v$.
Erklärung: As usual, $A - I_n$ defines a linear map

$$\begin{array}{rccc} T: & K^n_{\mathrm{col}} & \to & K^n_{\mathrm{col}} \\ & v & \mapsto & (A - I_n)\cdot v \end{array}$$

So,

$$n = \dim(K^n_{\mathrm{col}}) = \dim(\ker(T)) + \operatorname{rank}(T) = \dim(\ker(T)) + r.$$

If $r < n$, $\dim(\ker(T)) \geqslant 1$ so there exists some $v \in \ker(T) \setminus \{0\}$. Equivalently, there exists some non-zero $v \in V$ such that

$$A \cdot v - I_n \cdot v = 0 \Leftrightarrow A \cdot v = v.$$