Exercise 1.1.
Recall the definition of open set:

*A set $\Omega \subseteq \mathbb{R}^n$ is called **open** if for every point $x_0 \in \Omega$ $\exists r > 0$ s. t.*

$$B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\} \subseteq \Omega.$$

(a) Prove the following properties of open sets:

i) $\varnothing, \mathbb{R}^n$ are open;

ii) $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ open $\Rightarrow \Omega_1 \cap \Omega_2$ open;

iii) $\Omega_i \subseteq \mathbb{R}^n$ open $\forall i \in I \Rightarrow \bigcup_{i \in I} \Omega_i$ open (here I is an arbitrary index set).

Recall also:

*A set $A \subseteq \mathbb{R}^n$ is called **closed** if $\mathbb{R}^n \setminus A$ is open.*

(b) Prove the following properties of closed sets:

i) $\varnothing, \mathbb{R}^n$ are closed;

ii) $A_1, A_2 \subseteq \mathbb{R}^n$ closed $\Rightarrow A_1 \cup A_2$ closed;

iii) $A_i \subseteq \mathbb{R}^n$ closed $\forall i \in I \Rightarrow \bigcap_{i \in I} A_i$ closed (here I is again an arbitrary index set).

Exercise 1.2.

(a) Show that the intersection of infinitely many open sets is in general not open.

(b) Show that the union of infinitely many closed sets is in general not closed.

Exercise 1.3.

(a) Let A be a fixed subset of a set X. Determine the σ-algebra of subsets of X generated by $\{A\}$.

(b) Let X be an infinite set; let

$$\mathcal{A} = \{A \subset X : A \text{ or } A^c \text{ is finite}\}.$$

Prove that $\mathcal{A}$ is an algebra, but not a σ-algebra.

(c) Let X be an uncountable set[1]. Let

$$\mathcal{S} = \{E \subset X : E \text{ or } E^c \text{ is at most countable}\}.$$

Show that $\mathcal{S}$ is a σ-algebra and that $\mathcal{S}$ is generated by the one-point subsets of X.

[1] A set is uncountable if and only if its cardinality (which corresponds to the number of elements for finite sets) is bigger than that of the set of natural numbers.

Exercise 1.4.
Let X and Y be two sets and $f : X \to Y$ a map between them.

(a) If $\mathcal{B}$ is a σ-algebra on Y, show that

$$\{f^{-1}(E) : E \in \mathcal{B}\}$$

is a σ-algebra on X.

(b) If $\mathcal{A}$ is a σ-algebra on X, show that

$$\{E \subseteq Y : f^{-1}(E) \in \mathcal{A}\}$$

is a σ-algebra on Y.

Exercise 1.5.
Let X be a set and $\{A_n\}_{n=1}^{\infty}$ be a collection of subsets of X.

(a) Show the following:

$$\limsup_{n\to+\infty} A_n = \Big\{x \in X \;\Big|\; \forall N \geq 1, \exists n \geq N : x \in A_n\Big\}$$
$$\liminf_{n\to+\infty} A_n = \Big\{x \in X \;\Big|\; \exists N \geq 1, \forall n \geq N : x \in A_n\Big\}$$

(b) Show that $\liminf A_n \subset \limsup A_n$.

(c) Assume $X = \{1, 2, \ldots, 6\}^{\mathbb{N}}$ and $A_m = \{(x_n)_{n=1}^{\infty} \in X \mid x_m = 6\}$. Interpreting X as the possible outcomes of throwing a dice infinitely often and A_m as the subset of all outcomes where your m-th throw is a 6, give an interpretation of $\limsup A_m$ and $\liminf A_m$.

Exercise 1.6.
Let μ be a measure on a set X, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of X satisfying

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

Consider the set

$$E = \{x \in X : x \text{ belongs to } A_n \text{ for infinitely many } n\} = \limsup_{n\to+\infty} A_n,$$

show that $\mu(E) = 0$.