Exercise 2.1.
Prove that the system of elementary sets

$$\mathcal{A} := \{A \subset \mathbb{R}^n \mid A \text{ is the union of finitely many disjoint intervals}\}$$

is an algebra[1].

Exercise 2.2.
Let (X, Σ, μ) be a measure space. A subset $A \in \Sigma$ is called μ-*atom*, if it holds $\mu(A) > 0$ and for every $B \in \Sigma$ such that $B \subset A$, we have either $\mu(A \setminus B) = 0$ or $\mu(B) = 0$.

(a) Let A be a μ-atom and $B \in \Sigma$ such that $B \subset A$. Prove that either $\mu(B) = \mu(A)$ or $\mu(B) = 0$.

(b) Let $A \in \Sigma$ and assume $0 < \mu(A) < \infty$. Moreover, assume that for all $B \in \Sigma$ with $B \subset A$, it holds that either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Show that A is a μ-atom.

(c) Assume that μ is σ-finite, that is, there is a countable collection $\{S_j\} \subset \Sigma$ with $\mu(S_j) < \infty$ and $X = \bigcup_j S_j$. Show that for every μ-atom A, it holds $\mu(A) < \infty$.

Exercise 2.3.
Let X be an uncountable set and

$$\mathcal{B} := \big\{E \subset X \bigm| E \text{ or } E^c \text{ countable}\big\}.$$

Show that $\mu : \mathcal{B} \to [0, \infty]$ defined by

$$\mu(E) := \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{else} \end{cases}$$

is a pre-measure on $\mathcal{B}$[2].

Exercise 2.4.
Let X be a set and $\mu : \mathcal{P}(X) \to [0, \infty]$ a measure on X. Denote by $\mathcal{A}_\mu$ the σ-algebra of μ-measurable subsets of X. Let $B \subset X$ be an arbitrary subset.

(a) Denote by $\mu \llcorner B$ the restriction of μ to B defined by:

$$\forall A \subset X : \quad \mu \llcorner B(A) := \mu(A \cap B).$$

Show that $\mu \llcorner B$ is a measure.

(b) Show that $\mathcal{A}_\mu$ is a subset of the σ-algebra of $(\mu \llcorner B)$-measurable sets.

Exercise 2.5.
Let X, Y be two sets, $\mu : \mathcal{P}(X) \to [0, \infty]$ a measure on X and $f : X \to Y$ a map. How can we naturally define a "pushforward measure" $f_*\mu$ on Y? Prove that for such a measure, if $\mathcal{A}_\mu$ denotes the σ-algebra of μ-measurable sets in X, then the collection of sets[3]

$$f_*(\mathcal{A}_\mu) := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{A}_\mu\}$$

is a subset of the σ-algebra of $f_*\mu$-measurable subsets of Y.

[1] A sketch of the proof has been given in the lecture.
[2] See Definition 1.2.19 in the lecture notes.
[3] This is the σ-algebra introduced in Exercise 1.4 from Sheet 1.