Exercise 3.1.
Let μ be a measure on X and $A \subset X$ such that $\mu(A) < \infty$. Let $\{A_j\}_{j\in\mathbb{N}}$ be a countable family of μ-measurable subsets in X such that $A_j \subset A$ for all j. Assume that $\mu(A_j) \geq c_0 > 0$ for all $j \in \mathbb{N}$. Show that:

$$\mu\Big(\limsup_{j\to\infty} A_j\Big) \geq c_0.$$

Exercise 3.2.
Denote by λ the Lebesgue measure on $\mathbb{R}$. Let $E \subset [0,1]$ be a Lebesgue measurable set of strictly positive measure, i.e. $\lambda(E) > 0$. Show that for any $0 \leq \delta \leq \lambda(E)$, there exists a measurable subset of E having measure exactly δ.

Hint: Introduce the function which associates to $t \in [0,1]$ the measure of $[0,t] \cap E$. Is it continuous?

Exercise 3.3.
Let

$$\mathcal{A} = \{A \subset \mathbb{R}^n \mid A \text{ is the union of finitely many disjoint intervals}\}$$

denote the algebra of elementary sets in $\mathbb{R}^n$. Show that the volume function vol introduced in the lecture[1] for elementary sets is a pre-measure.

Remark: For $I = I_1 \times \ldots \times I_n$ an interval in $\mathbb{R}^n$, its volume is defined by

$$\mathrm{vol}(I) = \prod_{k=1}^{n} \mathrm{vol}(I_k),$$

where for an interval $I_k \subseteq \mathbb{R}$, $\mathrm{vol}(I_k)$ is the length of I_k.

Exercise 3.4.
Let X be any set with more than one element and consider the measure $\mu : \mathcal{P}(X) \to [0, +\infty]$ defined by:

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{else} \end{cases}.$$

Give an example of a non-μ-measurable subset.

[1] Definition 1.3.1 in the Lecture Notes.