Exercise 4.1.
Prove that the Lebesgue measure is invariant under translations, reflections and rotations, i.e. under all motions of the form

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi(x) = x_0 + Rx,$$

for $x_0 \in \mathbb{R}^n$ and $R \in O(n)$.

Hint: You may use the invariance of the Jordan measure, see Satz 9.3.2 in Struwe's lecture notes.

Exercise 4.2.
Show that every countable subset of $\mathbb{R}$ is a Borel set and has Lebesgue measure zero.

Exercise 4.3.
Show that the open ball $B(x,r) := \{y \in \mathbb{R}^n \mid |y-x| < r\}$ and the closed ball $\overline{B(x,r)} := \{y \in \mathbb{R}^n \mid |y-x| \leq r\}$ in $\mathbb{R}^n$ are Jordan measurable with Jordan measure $c_n r^n$, for some constant $c_n > 0$ depending only on n.

Exercise 4.4.

(a) Let $A \subset \mathbb{R}$ be a subset with Lebesgue measure $\mathcal{L}^1(A) > 0$. Show that there exists a subset $B \subset A$ which is **not** $\mathcal{L}^1$-measurable.

(b) Find an example of a countable, pairwise disjoint collection $\{E_k\}_k$ of subsets in $\mathbb{R}$, such that:

$$\mathcal{L}^1\Big(\bigcup_{k=1}^{\infty} E_k\Big) < \sum_{k=1}^{\infty} \mathcal{L}^1(E_k).$$

Exercise 4.5.
Fix some $0 < \beta < 1/3$ and define $I_1 = [0,1]$. For every $n \geq 1$, let $I_{n+1} \subset I_n$ be the collection of intervals obtained removing from every interval in I_n its centered open subinterval of length β^n. Then define by $C_\beta = \bigcap_{n=1}^{\infty} I_n$, the *fat Cantor set* corresponding to β.

Show that:

(a) C_β is Lebesgue measurable with measure $\mathcal{L}^1(C_\beta) = 1 - \frac{\beta}{1-2\beta}$.

(b) C_β is not Jordan measurable. Indeed it holds $\underline{\mu}(C_\beta) = 0$ and $\overline{\mu}(C_\beta) = 1 - \frac{\beta}{1-2\beta} > 0$.