Exercise 5.1.
The goal of this exercise is to show that the Cantor triadic set C is uncountable. For that, recall quickly the construction of C: Every $x \in [0,1]$ can be expanded in base 3, i.e., can be written as $x = \sum_{i=1}^{\infty} d_i(x)3^{-i}$ for $d_i(x) \in \{0,1,2\}$. The set C is then defined as the set of those $x \in [0,1]$ that do not have any digit 1 in their 3-expansion, i.e.:,

$$C := \{x \in [0,1] \mid d_i(x) \in \{0,2\}, \forall i \in \mathbb{N}\}.$$

Now, the Cantor-Lebesgue function F is defined by

$$F : C \to [0,1], \quad F\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) := \sum_{i=1}^{\infty} \frac{a_i}{2^{i+1}}.$$

(a) Show that $F(0) = 0$ and $F(1) = 1$.

(b) Show that F is well-defined and continuous on C.

(c) Show that F is surjective.

(d) Conclude that C is uncountable.

Exercise 5.2.
Let E be the collection of all numbers in $[0,1]$ whose decimal expansion with respect to the basis 10 has no sevens appearing.

Recall that some decimals have two possible expansions. We are taking the convention that no expansion should be identically zero from some digit onward; for example $\frac{27}{100}$ should be written as $0,269999....$ and not as $0,27$.

Prove that E is a Lebesgue measurable set and determine its Lebesgue measure.

Exercise 5.3.
Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz with constant L. Let $A \subset \mathbb{R}^n$ and $0 \leq s < +\infty$. Show that

$$\mathcal{H}^s(f(A)) \leq L^s \mathcal{H}^s(A).$$

Exercise 5.4.
Let C denote the Cantor set as defined in the lecture. Show that it holds

$$\dim_{\mathcal{H}}(C) = \frac{\ln(2)}{\ln(3)} =: s,$$

and that $2^{-s-1} \leq \mathcal{H}^s(C) \leq 2^{-s}$.