Exercise 6.1.
For $s \geq 0$ and $\emptyset \neq A \subset \mathbb{R}^n$, we define

$$\mathcal{H}^s_\infty(A) := \inf\left\{\sum_{k\in I} r_k^s \mid A \subset \bigcup_{k\in I} B(x_k, r_k),\ r_k > 0\right\},$$

where the set of indices I is at most countable. One can check that $\mathcal{H}^s_\infty$ is a measure. Prove that $\mathcal{H}^{1/2}_\infty$ is not Borel on $\mathbb{R}$.

Remark. Note that the definition of $\mathcal{H}^s_\infty$ coincides with Definition 1.8.1 in the Lecture Notes for $\delta = \infty$.

Exercise 6.2.
Prove the following claims.

(a) The Lebesgue measure $\mathcal{L}^n$ is a Radon measure on $\mathbb{R}^n$.

(b) The Hausdorff measure $\mathcal{H}^s$ is not a Radon measure for $s < n$, but it is a Radon measure for $s \geq n$.

(c) If μ is a Radon measure and $A \subset \mathbb{R}^n$ is μ-measurable, then $\mu \llcorner A$ given by

$$(\mu \llcorner A)(B) := \mu(A \cap B), \quad \forall\ B \subset \mathbb{R}^n$$

is a Radon measure as well.

Exercise 6.3.
Given any subset $A \subset \mathbb{R}^n$, show that $\dim_{\mathcal{H}}(A) = \sup\{t \geq 0 \mid \mathcal{H}^t(A) = +\infty\}$.

Exercise 6.4.
Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuous injective curve. We define the arc length of γ as

$$L(\gamma) := \sup\left\{\sum_{i=1}^N d(\gamma(t_{i-1}), \gamma(t_i)) \mid N \in \mathbb{N},\ a \leq t_0 \leq t_1 \leq \ldots \leq t_N \leq b\right\}.$$

Show that $\mathcal{H}^1(\mathrm{Im}(\gamma)) = \frac{1}{2}L(\gamma)$.

Exercise 6.5.
Consider the continuous function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}.$$

(a) Show that the graph of f has infinite length as a curve, and therefore the set

$$A := \{(x, f(x)) \mid x \in [0, 1]\}$$

has $\mathcal{H}^1(A) = \infty$.
Hint: use Exercise 6.4 to relate the length with the $\mathcal{H}^1$ measure.

(b) Show that $\mathcal{H}^s(A) = 0$ if $s > 1$.

(c) Conclude that $\dim_{\mathcal{H}}(A) = 1$.