Exercise 7.1.
Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Show that the following statements are equivalent.

(i) $f^{-1}(U)$ is μ-measurable for every open set $U \subset \mathbb{R}$.

(ii) $f^{-1}(B)$ is μ-measurable for every Borel set $B \subset \mathbb{R}$.

(iii) $f^{-1}((-\infty, a))$ is μ-measurable for every $a \in \mathbb{R}$.

Exercise 7.2.
Let (X, μ, Σ) be a measure space and $f, g : X \to \mathbb{R}$ two measurable functions on X. Show that the sets $\{x \mid f(x) = g(x)\}$ and $\{x \mid f(x) < g(x)\}$ are measurable.

Exercise 7.3.
A function $g\colon \mathbb{R} \to \mathbb{R}$ is called Borel measurable, if for every open set $U \subset \mathbb{R}$ the set $g^{-1}(U)$ is a Borel set. Let (X, μ, Σ) be a measure space, let $f\colon X \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ be functions with f μ-measurable and g Borel measurable. Show that $g \circ f$ is μ-measurable.

Exercise 7.4.
In this exercise, we construct a set which is Lebesgue measurable, but not Borel and use the construction to give an example of a continuous $G : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $F : \mathbb{R} \to \mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

(a) Let $h : [0, 1] \to [0, 1]$ be the Cantor function, which is the unique monotonically increasing extension of the function $C \to [0, 1]$ seen in Exercise 5.1, where $C \subset [0, 1]$ is the Cantor set. Define $g : [0, 1] \to [0, 2]$ by $g(x) := h(x) + x$. Show that g is strictly monotone and a homeomorphism.

(b) Show that $\mathcal{L}^1(g(C)) = 1$.

Hint: Use the natural decomposition of $[0, 1] \setminus C$ to deduce the result.

(c) Use Exercise 4.4 (a) to find a non-measurable subset $E \subset g(C)$ and define $A := g^{-1}(E)$. Show that A is a Lebesgue zero set and thus Lebesgue measurable.

(d) Show that A is not a Borel set.

Hint: Otherwise, the preimage of A with respect to continuous maps would necessarily be Borel and thus Lebesgue measurable as well.

(e) Find appropriate F, G as outlined above such that $F \circ G$ is not Lebesgue measurable, using the sets and functions introduced in the previous subtasks.

Exercise 7.5.
Let μ be a Borel measure on $\mathbb{R}$ and $f\colon [a, b] \to \mathbb{R}$ continuous μ-almost everywhere (i.e., the set of points where f is not continuous is a set of μ-measure zero). Show that f is μ-measurable.

Exercise 7.6.
Let μ be a Borel measure on $\mathbb{R}$. Show that every monotone function $f\colon [a, b] \to \mathbb{R}$ is μ-measurable.