Exercise 8.1.
Prove Littlewood's first principle: Let μ be a Radon measure on $\mathbb{R}^n$ and $E \subseteq \mathbb{R}^n$ a μ-measurable set with $\mu(E) < \infty$. Then for every $\varepsilon > 0$ there exists an elementary set F such that $\mu(E \triangle F) < \varepsilon$.

Exercise 8.2.
Let $f_k \colon \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-measurable functions, for $k \in \mathbb{N}$. Assume that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \mid |f_k(x) - f_{k+1}(x)| > 2^{-k}\}) < 2^{-k}$$

for all $k \in \mathbb{N}$. Show that the limit $\lim_{k\to\infty} f_k(x)$ exists almost everywhere.

Exercise 8.3.
Let μ be a measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ be μ-measurable. Let $f \colon \Omega \to \overline{\mathbb{R}}$ be a finite, μ-measurable function, and $(f_k)_{k\in\mathbb{N}}$ a sequence of μ-measurable functions $f_k \colon \Omega \to \overline{\mathbb{R}}$ with the following property: Every subsequence $(f_{k_j})_{j\in\mathbb{N}}$ contains a subsequence that converges to f in measure μ.

(a) Show that the whole sequence $(f_k)_{k\in\mathbb{N}}$ converges to f in measure μ.

(b) Show that the analogous statement from (a) is not true, if we assume only pointwise convergence μ-almost everywhere.

Exercise 8.4.
Counterexample to $\varepsilon = 0$ in Lusin's Theorem: Find an example of a $\mathcal{L}^1$-measurable function $f : [0,1] \to \mathbb{R}$ such that for every $\mathcal{L}^1$-measurable set $M \subset [0,1]$ with $\mathcal{L}^1(M) = 1$, the restriction $f|_M : M \to \mathbb{R}$ is discontinuous in all but finitely many points of M.

Hint: You may use that there exists a Lebesgue measurable subset $A \subset [0,1]$ such that

$$\mathcal{L}^1(U \cap A) \cdot \mathcal{L}^1(U \cap A^c) > 0$$

for all nonempty open subsets $U \subset [0,1]$. Such a set A can be constructed using the Cantor set (see Remark 1.6.2).

Exercise 8.5.
Counterexample to $\delta = 0$ in Egoroff's Theorem: Find an example of a sequence of $\mathcal{L}^1$-measurable functions $f_k : [0,1] \to \overline{\mathbb{R}}$ that converges pointwise almost everywhere to a $\mathcal{L}^1$-measurable ($\mathcal{L}^1$-almost everywhere finite) function $f : [0,1] \to \overline{\mathbb{R}}$, but for every compact $F \subset [0,1]$ with $\mathcal{L}^1(F) = \mathcal{L}^1([0,1])$ the convergence on F is not uniform.

Exercise 8.6.
Let μ be a measure on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ a μ-measurable set and $f : \Omega \to [0, \infty]$ a μ-measurable function. Consider the sets $A_j \subseteq \Omega$ from Theorem 2.2.6 of the Lecture Notes, defined so that the sequence of functions

$$f_k = \sum_{j=1}^{k} \frac{1}{j} \chi_{A_j}$$

converges pointwise to f. Show that if f is bounded, then f_k converge uniformly to f, that is,

$$\sup_{x \in \Omega} |f(x) - f_k(x)| \longrightarrow 0 \text{ as } k \to \infty.$$