Exercise 9.1.

In this exercise, we prove the linearity, monotonicity and well-definedness of the integral of simple functions, see Definitions 3.1.2 and 3.1.3 in the Lecture Notes. These results are essential to derive the corresponding properties of the general integral.

Remark. Throughout the exercise, we assume that all simple functions introduced are at least μ-integrable.

(a) Let f, g be two μ-measurable simple functions with values $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ (see Definition 3.1.1. in the Lecture Notes). Show that there exist μ-measurable, disjoint sets $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, such that

$$f = \sum_{n\in\mathbb{N}} a_n \chi_{A_n}, \quad g = \sum_{n\in\mathbb{N}} b_n \chi_{B_n},$$

and prove that the sets and values can be chosen in such a way that $A_n = B_n$ holds for all $n \in \mathbb{N}$.

(b) Assume that $f = \sum_{n\in\mathbb{N}} a_n \chi_{A_n}$, where $\{a_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{R}}$ is a sequence of values (not necessarily different from each other) and $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint, μ-measurable subsets. Prove that

$$\int f d\mu = \sum_{n\in\mathbb{N}} a_n \mu(A_n).$$

(c) Let f, g be μ-measurable simple functions such that $f \le g$ μ-almost everywhere. Then it holds

$$\int f d\mu \le \int g d\mu.$$

(d) Assume f, g are μ-summable simple functions (see Definition 3.1.8) and $a, b \in \mathbb{R}$. Show that $af + bg$ is a μ-summable simple function and

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu.$$

(e) Let f be a μ-integrable simple function. Prove that

$$\underline{\int} f d\mu = \overline{\int} f d\mu = \int f d\mu,$$

where the last integral is understood in the sense of integrals for simple functions, see Definitions 3.1.2 and 3.1.3 in the Lecture Notes.

Exercise 9.2.

(a) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of μ-measurable functions on a μ-measurable set $\Omega \subset \mathbb{R}^n$. Show that the series $\sum_{k=1}^{\infty} f_k(x)$ converges μ-almost everywhere, if

$$\sum_{k=1}^{\infty} \int_{\Omega} |f_k| d\mu < \infty.$$

(b) Let $\{r_k\}_{k\in\mathbb{N}}$ be an ordering of $\mathbb{Q}\cap[0,1]$ and $(a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be such that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. Show that $\sum_{k=1}^{\infty} a_k|x-r_k|^{-1/2}$ is absolutely convergent for almost every $x \in [0,1]$ (with respect to the Lebesgue measure).

Exercise 9.3.
Find an example of a continuous bounded function $f\colon [0,\infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$ and

$$\int_0^{\infty} |f(x)|^p dx = \infty \ ,$$

for all $p > 0$.

Exercise 9.4.
Let $f, g : \Omega \to \overline{\mathbb{R}}$ be μ-summable functions and assume that

$$\int_A f d\mu \leq \int_A g d\mu$$

for all μ-measurable subsets $A \subset \Omega$. Show that $f \leq g$ μ-almost everywhere. Moreover, conclude that, if

$$\int_A f d\mu = \int_A g d\mu$$

for all μ-measurable subsets $A \subset \Omega$, then $f = g$ μ-almost everywhere.

Exercise 9.5.
Let $f_n\colon \mathbb{R} \to \overline{\mathbb{R}}$ be Lebesgue measurable functions. Find examples for the following statements.

(a) $f_n \to 0$ uniformly, but not $\int |f_n| dx \to 0$.

(b) $f_n \to 0$ pointwise and in measure, but neither $f_n \to 0$ uniformly nor $\int |f_n| dx \to 0$.

(c) $f_n \to 0$ pointwise, but not in measure.

Exercise 9.6.
Let $f : \Omega \to [0,\infty]$ be μ-measurable. Prove the following facts:

(a) If $\int_\Omega f d\mu = 0$, then $f = 0$ μ-almost everywhere.

(b) If $\int_\Omega f d\mu < +\infty$, then $f < +\infty$ μ-almost everywhere.