Exercise 10.1.
Let $f : \Omega \to \overline{\mathbb{R}}$ be μ-summable and $\Omega_1 \subseteq \Omega$ be μ-measurable. Show that $f_1 := f|_{\Omega_1}$ and $f\chi_{\Omega_1}$ are μ-summable on Ω_1 and Ω respectively, and that

$$\int_{\Omega_1} f_1 \, d\mu = \int_\Omega f \, \chi_{\Omega_1} d\mu.$$

Exercise 10.2.
Show that if $f : \Omega \to \overline{\mathbb{R}}$ is a μ-summable function and $\Omega_1 \subseteq \Omega$ has $\mu(\Omega_1) = 0$, then

$$\int_{\Omega_1} f \, d\mu = 0.$$

Exercise 10.3.
By applying Lebesgue's Theorem to the counting measure on $\mathbb{N}$, show that

$$\lim_{n\to\infty} n \sum_{i=1}^{\infty} \sin\left(\frac{2^{-i}}{n}\right) = 1.$$

Exercise 10.4.
Let λ be the Lebesgue measure on $\mathbb{R}$ and f a nonnegative summable function on $(\mathbb{R}, \lambda)$. Show that the following equality of Lebesgue integrals holds:

$$\int_{\mathbb{R}} f d\lambda = \int_0^{+\infty} \lambda(\{f > s\}) ds.$$

Hint: In a first instance prove the equality when f is a simple function; in this case, make a picture of f and of the function $s \mapsto \lambda(\{f > s\})$ and interpret the two sides according to the definition of the Lebesgue integral.

Exercise 10.5.
For all $n \in \mathbb{N}$, let $f_n \colon [0, 1] \to \mathbb{R}$ be defined by:

$$f_n(x) = \frac{n\sqrt{x}}{1 + n^2x^2}.$$

Prove that:

(a) $f_n(x) \le \frac{1}{\sqrt{x}}$ on $(0, 1]$ for all $n \ge 1$.

(b) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$

Exercise 10.6.
Let $\Omega \subseteq \mathbb{R}^n$ be μ-measurable with $\mu(\Omega) < +\infty$ and let $\{f_j\}$ be a sequence of μ-summable $\overline{\mathbb{R}}$-valued functions such that $f_j \to f$ uniformly in Ω. Show that f is μ-summable and

$$\lim_{j\to\infty} \int_\Omega f_j \, d\mu = \int_\Omega f \, d\mu.$$