Exercise 11.1.
Compute the limit

$$\lim_{n\to\infty}\int_a^{+\infty}\frac{n}{1+n^2x^2}\,dx$$

for every $a\in\mathbb{R}$.

Hint: recall that $\arctan x$ is a primitive of $\frac{1}{1+x^2}$.

Exercise 11.2.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$ be μ-measurable and $f\colon\Omega\to[0,+\infty]$ be μ-summable. For all μ-measurable subsets $A\subset\Omega$ define (see Section 3.5 in the Lecture Notes)

$$\nu(A)=\int_A f\,d\mu.$$

(a) Prove that ν is a pre-measure on the σ-algebra of μ-measurable sets, hence we can define its Carathéodory-Hahn extension $\nu\colon\mathcal{P}(\Omega)\to[0,+\infty]$.

(b) Show that ν is a Radon measure.

(c) Prove that ν is absolutely continuous with respect to μ.

Exercise 11.3.
(a) Let $f\colon[a,+\infty)\to\mathbb{R}$ be a locally bounded function and locally Riemann integrable. Then f is $\mathcal{L}^1$-summable if and only if f is absolutely Riemann integrable in the generalized sense (namely $\mathcal{R}\int_a^\infty|f(x)|dx=\lim_{j\to\infty}\mathcal{R}\int_a^j|f(x)|dx$ exists and it is finite) and in this case

$$\int_{[a,+\infty)}f(x)d\mathcal{L}^1=\mathcal{R}\int_a^\infty f(x)dx=\lim_{j\to+\infty}\mathcal{R}\int_a^j f(x)dx.$$

(b) Let $f\colon[0,+\infty)\to\mathbb{R}$ be the function $f(x)=\frac{\sin x}{x}$, which is locally bounded and locally Riemann integrable. Show that f is Riemann integrable, i.e. $\mathcal{R}\int_0^\infty f(x)dx<+\infty$ but not absolutely Riemann integrable, i.e. $\mathcal{R}\int_0^\infty|f(x)|dx=\infty$. Hence f is not $\mathcal{L}^1$-summable.

Exercise 11.4.
Construct a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n:[0,1]\to\mathbb{R}$ such that

- the f_n are Riemann integrable and $f_n\le f_{n+1}$ (monotonically increasing sequence);
- $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to a function f which is NOT Riemann integrable (so $f_n(x)\to f(x)$ for all $x\in[0,1]$).

Check that Beppo Levi's Theorem holds for the constructed sequence.

Exercise 11.5.
(a) Let μ be a Radon measure on $\mathbb{R}^n$ and let $\Omega\subset\mathbb{R}^n$ be a μ-measurable subset. Consider a function $f\colon\Omega\times(a,b)\to\mathbb{R}$, for some interval $(a,b)\subset\mathbb{R}$, such that:

- the map $x\mapsto f(x,y)$ is μ-summable for all $y\in(a,b)$;

- the map $y \mapsto f(x,y)$ is differentiable in (a,b) for every $x \in \Omega$;
- there is a μ-summable function $g\colon \Omega \to [0,\infty]$ such that $\sup_{a<y<b}|\frac{\partial f}{\partial y}(x,y)| \leq g(x)$ for all $x \in \Omega$.

Then $y \mapsto \int_\Omega f(x,y)d\mu(x)$ is differentiable in (a,b) with

$$\frac{d}{dy}\left(\int_\Omega f(x,y)d\mu(x)\right) = \int_\Omega \frac{\partial f}{\partial y}(x,y)d\mu(x)$$

for all $y \in (a,b)$.

(b) Compute the integral

$$\phi(y) := \int_{(0,\infty)} e^{-x^2-y^2/x^2} d\mathcal{L}^1(x)$$

for all $y > 0$.

Hint: use part (a) to obtain that ϕ solves the Cauchy problem

$$\begin{cases} \phi'(y) = -2\phi(y) & \text{for } y > 0 \\ \lim_{y\to 0^+}\phi(y) = \sqrt{\pi}/2. \end{cases}$$

Exercise 11.6.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$ and $f, f_k : \Omega \to \overline{\mathbb{R}}$ μ-summable functions.

(a) Show that Vitali's Theorem implies Dominated Convergence Theorem.

(b) Let $\Omega = [0,1]$ and $\mu = \mathcal{L}^1$. Give an example in which Vitali's Theorem can be applied but Dominated Convergence Theorem cannot, i.e., a dominating function does not exist.

Hint: look at the functions $f_n^k(x) = \frac{1}{x}\chi_{[\frac{n+k-1}{n2^{n+1}}, \frac{n+k}{n2^{n+1}})}(x)$ for $n \in \mathbb{N}$, $1 \leq k \leq n$.