Exercise 12.1.
The goal of this exercise is to compute the following Riemann integral:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{a\to\infty}\int_0^a \frac{\sin x}{x}\,dx.$$

(a) Show that the function $\Phi : (0,\infty) \to \mathbb{R}$,

$$\Phi(t) = \int_0^\infty e^{-tx}\frac{\sin x}{x}\,dx,$$

is well-defined and differentiable everywhere.

(b) Compute $\Phi'(t)$ for $t \in (0,\infty)$.

(c) Compute $\Phi(t)$ for $t \in (0,\infty)$.

(d) Show that the convergence

$$\int_0^a e^{-tx}\frac{\sin x}{x}\,dx \xrightarrow{a\to\infty} \int_0^\infty e^{-tx}\frac{\sin x}{x}\,dx$$

is uniform in $t > 0$.

Hint: this part is technically more difficult. It is not true that $\int_a^\infty \left|e^{-tx}\frac{\sin x}{x}\right|\,dx$ converges to zero uniformly in t as $a \to \infty$. Here one has to use the cancellations of the integral, for example by seeing that

$$\sum_{k=m}^\infty \left|\int_{2k\pi}^{2(k+1)\pi} e^{-tx}\frac{\sin x}{x}\,dx\right|$$

converges to zero as $m \to \infty$ uniformly in t.

(e) Conclude that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Exercise 12.2.
Let $1 \le p < \infty$. Show that if $\varphi \in L^p(\mathbb{R}^n)$ and φ is uniformly continuous, then

$$\lim_{|x|\to\infty} \varphi(x) = 0.$$

Exercise 12.3.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set.

(a) (Generalized Hölder inequality) Consider $1 \le p_1, \dots, p_k \le \infty$ such that $\frac{1}{r} = \sum_{i=1}^k \frac{1}{p_i} \le 1$. Show that, given functions $f_i \in L^{p_i}(\Omega,\mu)$ for $i = 1,\dots,k$, it holds $\prod_{i=1}^k f_i \in L^r(\Omega,\mu)$ and

$$\left\|\prod_{i=1}^k f_i\right\|_{L^r} \le \prod_{i=1}^k \|f_i\|_{L^{p_i}}.$$

(b) Prove that, if $\mu(\Omega) < +\infty$, then $L^s(\Omega, \mu) \subseteq L^r(\Omega, \mu)$ for all $1 \le r < s \le +\infty$.

(c) Show that the inclusion in part (b) is strict for all $1 \le r < s \le +\infty$.

Exercise 12.4.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$. Consider a function $f \colon \Omega \to \overline{\mathbb{R}}$ such that $fg \in L^1(\Omega, \mu)$ for all $g \in L^p(\Omega, \mu)$. Prove that $f \in L^q(\Omega, \mu)$ for all $q \in [1, p')$, where $p' = \frac{p}{p-1}$ is the conjugate of p.

Exercise 12.5.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set.

(a) Show that any $f \in \bigcap_{p \in \mathbb{N}^*} L^p(\Omega, \mu)$ with $\sup_{p \in \mathbb{N}^*} \|f\|_{L^p} < +\infty$ lies in $L^\infty(\Omega, \mu)$.
Hint. Tchebychev' inequality.

(b) Show that if $\mu(\Omega) < +\infty$, then for any f as in part (a) we have that $\|f\|_{L^\infty} = \lim\limits_{p \to \infty} \|f\|_{L^p}$.

(c) Find $f \in \bigcap_{p \in \mathbb{N}} L^p(\Omega, \mu)$, where $\mu(\Omega) < +\infty$, with $f \notin L^\infty(\Omega, \mu)$, i.e., show that the result from part (a) does not hold true without the assumption $\sup\limits_{p \in \mathbb{N}} \|f\|_{L^p} < +\infty$.

Exercise 12.6.
Let $(x_{n,m})_{(n,m) \in \mathbb{N}^2} \subset [0, +\infty]$ be a sequence parametrized by $\mathbb{N}^2$. Show that

$$\sum_{(n,m) \in \mathbb{N}^2} x_{n,m} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_{n,m} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x_{n,m}.$$

Remark. Given a sequence $(x_\alpha)_{\alpha \in A} \subset [0, +\infty]$ parametrized by an arbitrary set A, we define

$$\sum_{\alpha \in A} x_\alpha := \sup_{F \subset A \text{ finite}} \sum_{\alpha \in F} x_\alpha.$$