Exercise 13.1.
Let $f \in L^p(\mathbb{R}, \lambda)$, where λ is the Lebesgue measure. By means of Fubini's Theorem, show that the following equality holds:

$$\int_{\mathbb{R}} |f(x)|^p dx = p \int_0^\infty y^{p-1} \lambda(\{x \in \mathbb{R} : |f(x)| \geq y\}) dy.$$

Hint: $|f(x)|^p = \int_0^{|f(x)|} p y^{p-1} dy$.

Remark. Compare with Exercise 10.4. In that case there was an underlying Fubini-type argument in the proof. This time we can use Fubini's Theorem and get a straightforward proof.

Exercise 13.2.
Define the function $f : [0,1]^2 \to \mathbb{R}$ as

$$f(x,y) := \begin{cases} y^{-2} & \text{if } 0 < x < y < 1, \\ -x^{-2} & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Is this function summable with respect to the Lebesgue measure?

Exercise 13.3.
Let $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^n)$ and, for all $h \in \mathbb{R}^n$, consider the function $\tau_h \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\tau_h(x) = x + h$. Show that

$$\|f \circ \tau_h - f\|_{L^p} \to 0 \qquad \text{as } h \to 0.$$

Hint: use the density of continuous and compactly supported functions in L^p (Theorem 3.7.15 in the Lecture Notes).

Exercise 13.4.
We say that a family $(\varphi_\varepsilon)_{\varepsilon>0}$ of functions in $L^1(\mathbb{R}^n)$ is an *approximate identity* if:

1. $\varphi_\varepsilon \geq 0$ and $\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = 1$ for all $\varepsilon > 0$;
2. for all $\delta > 0$ we have that $\int_{\{|x| \geq \delta\}} \varphi_\varepsilon(x) dx \to 0$ as $\varepsilon \to 0$.

(a) Given $\varphi \in L^1(\mathbb{R}^n)$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi(x) dx = 1$, define $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\varepsilon^{-1} x)$ for all $\varepsilon > 0$. Show that $(\varphi_\varepsilon)_{\varepsilon>0}$ is an approximate identity.

Let $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^n)$ be an approximate identity. Show that the following statements hold.

(b) If $f \in L^\infty(\mathbb{R}^n)$ is continuous at $x_0 \in \mathbb{R}^n$, then $f * \varphi_\varepsilon$ is continuous and $(f * \varphi_\varepsilon)(x_0) \to f(x_0)$ as $\varepsilon \to 0^+$.

(c) If $f \in L^\infty(\mathbb{R}^n)$ is uniformly continuous, then $f * \varphi_\varepsilon \xrightarrow{L^\infty} f$ as $\varepsilon \to 0^+$.

(d) If $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^n)$, then $f * \varphi_\varepsilon \xrightarrow{L^p} f$ as $\varepsilon \to 0^+$.

Hint: use Hölder's inequality and keep in mind Exercise 13.3 and part (b).

Exercise 13.5.
Compute the following limits:

(a)
$$\lim_{n\to\infty}\int_0^1 \frac{1+nx}{(1+x)^n}\,dx.$$

(b)
$$\lim_{n\to\infty}\int_0^1 \frac{x\log x}{1+n^2x^2}\,dx.$$

Exercise 13.6.
Let $I=[0,1]$ and consider the function

$$f : I^3 \to [0,\infty], \quad f(x,y,z) := \begin{cases} \frac{1}{\sqrt{|y-z|}}, & \text{if } y \neq z, \\ \infty, & \text{if } y = z. \end{cases}$$

Show that $f \in L^1(I^3, \mathcal{L}^3)$.

Exercise 13.7.
Find a sequence of Lebesgue-measurable functions $f_n : [0,1] \to \mathbb{R}$ such that $\{f_n(x)\}_{n\in\mathbb{N}}$ is unbounded for any $x \in [0,1]$ but $f_n \to 0$ in measure.