Exercise 1.1.
Recall the definition of open set:

*A set $\Omega \subseteq \mathbb{R}^n$ is called **open** if for every point $x_0 \in \Omega$ $\exists r > 0$ s. t.*

$$B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\} \subseteq \Omega.$$

(a) Prove the following properties of open sets:

i) $\varnothing, \mathbb{R}^n$ are open;
ii) $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ open $\Rightarrow \Omega_1 \cap \Omega_2$ open;
iii) $\Omega_i \subseteq \mathbb{R}^n$ open $\forall i \in I \Rightarrow \bigcup_{i\in I} \Omega_i$ open (here I is an arbitrary index set).

Solution:

i) For $\varnothing$ there is nothing to prove; for $\mathbb{R}^n$, it suffices to take $r = 1$ for each $x_0 \in \mathbb{R}^n$, because $B_1(x_0) \subseteq \mathbb{R}^n$.

ii) Let $x_0 \in \Omega_1 \cap \Omega_2$ and let r_i such that $B_{r_i}(x_0) \subseteq \Omega_i$ for $i = 1, 2$. Then set r to be the minimum of r_1 and r_2, so that $r > 0$ and

$$B_r(x_0) \subseteq B_{r_i}(x_0) \subseteq \Omega_i$$

for each $i \in \{1, 2\}$, and therefore $B_r(x_0)$ lies in the intersection.

iii) Let $\Omega := \bigcup_{i\in I} \Omega_i$ and $x_0 \in \Omega$. Therefore $\exists i \in I$ such that $x_0 \in \Omega_i$ and thus we can pick $r > 0$ with $B_r(x_0) \subseteq \Omega_i \subseteq \Omega$.

Recall also:

*A set $A \subseteq \mathbb{R}^n$ is called **closed** if $\mathbb{R}^n \setminus A$ is open.*

(b) Prove the following properties of closed sets:

i) $\varnothing, \mathbb{R}^n$ are closed;
ii) $A_1, A_2 \subseteq \mathbb{R}^n$ closed $\Rightarrow A_1 \cup A_2$ closed;
iii) $A_i \subseteq \mathbb{R}^n$ closed $\forall i \in I \Rightarrow \bigcap_{i\in I} A_i$ closed (here I is again an arbitrary index set).

Solution: All the properties follow from the corresponding properties of open sets and the laws of De Morgan. More precisely, i) is trivial, ii) follows from

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$$

and iii) follows from

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

Exercise 1.2.
(a) Show that the intersection of infinitely many open sets is in general not open.

Solution: Let $n = 1$ and consider the open sets $\Omega_k = \left(-\frac{1}{k}, +\infty\right) \subset \mathbb{R}$ for $k = 1, 2, \ldots$. Their intersection is clearly the interval $[0, +\infty)$, which is of course not open because it contains no interval around 0. Note that even for countably many open sets the statement is false.

(b) Show that the union of infinitely many closed sets is in general not closed.

Solution: We can take as an example the complements of the above sets: $A_k = (-\infty, -\frac{1}{k}]$ for $k = 1, 2, \ldots$. Their union is the open interval $(-\infty, 0)$, which is not closed because its complement $[0, +\infty)$, as we have seen, is not open.

Exercise 1.3.

(a) Let A be a fixed subset of a set X. Determine the σ-algebra of subsets of X generated by $\{A\}$.

Solution: The σ-algebra generated by $\{A\}$ necessarily contains the following elements:

$$\emptyset, A, A^c, X.$$

Due to the collection $\{\emptyset, A, A^c, X\}$ already being closed under taking complements and unions of sets, this is the σ-algebra generated by $\{A\}$. □

(b) Let X be an infinite set; let

$$\mathcal{A} = \{A \subset X : A \text{ or } A^c \text{ is finite}\}.$$

Prove that $\mathcal{A}$ is an algebra, but not a σ-algebra.

Solution: Clearly, we have that $X \in \mathcal{A}$ and $\mathcal{A}$ is closed under complement. Thus we only need to check that $\mathcal{A}$ is closed under finite union. Let $A, B \in \mathcal{A}$. If A and B are finite, then $A \cup B$ is finite and thus $A \cup B \in \mathcal{A}$. Otherwise at least one between A^c and B^c is finite and therefore $(A \cup B)^c = A^c \cap B^c$ is finite, which implies again $A \cup B \in \mathcal{A}$.

We now prove that $\mathcal{A}$ is not a σ-algebra (for every infinite set X). Indeed, let $Y = \{a_n\}_n \subset X$ be a countable subset such that Y^c is infinite, and define $A_n = \{a_n\}$ for all n. Note that $A_n \in \mathcal{A}$, since it is finite. On the other hand $\cup_{n=1}^{\infty} A_n = Y$ is infinite with Y^c infinite, hence it is not contained in $\mathcal{A}$. This proves that $\mathcal{A}$ is not a σ-algebra because it is not closed under countable union. □

(c) Let X be an uncountable set; let

$$\mathcal{S} = \{E \subset X : E \text{ or } E^c \text{ is at most countable}\}.$$

Show that $\mathcal{S}$ is a σ-algebra and that $\mathcal{S}$ is generated by the one-point subsets of X.

Solution: Firstly, let us show that $\mathcal{S}$ is a σ-algebra. Clearly, $\emptyset$ and X belong to $\mathcal{S}$. Moreover, it is easy to see that $\mathcal{S}$ is closed under taking complements. Let therefore $\{A_k\} \subset \mathcal{S}$. If all sets $\{A_k\}$ are at most countable, then so is $\cup_{k=1}^{\infty} A_k$, implying that the union again belongs to $\mathcal{S}$. Otherwise, A_m is uncountable for some m, therefore A_m^c is at most countable. Due to the inclusion

$$\left(\bigcup_{k=1}^{\infty} A_k \right)^c \subset A_m^c,$$

the complement of $\cup_{k=1}^{\infty} A_k$ is at most countable and therefore

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{S}.$$

It remains to show that $\mathcal{S}$ is generated by the one-point subsets of X. By the definition of $\mathcal{S}$, all one-point subsets belong to $\mathcal{S}$. In addition, for every A in $\mathcal{S}$, either A or its complement can be expressed as a countable union of one-point subsets. Consequently, every element in $\mathcal{S}$ can be obtain from the one-point subsets using unions and complements. □

Exercise 1.4.

Let X and Y be two sets and $f : X \to Y$ a map between them.

(a) If $\mathcal{B}$ is a σ-algebra on Y, show that

$$\{f^{-1}(E) : E \in \mathcal{B}\}$$

is a σ-algebra on X.

Solution: Let $\mathcal{A}$ be the collection of sets defined in this way. Observe that $X = f^{-1}(Y)$, so that $X \in \mathcal{A}$ and

$$(f^{-1}(B))^c = \{x \in X \mid f(x) \notin B\} = \{x \in X \mid f(x) \in B^c\} = f^{-1}(B^c),$$

so that $\mathcal{A}$ is closed with respect to complements. Finally, given a sequence $(A_i) \subset \mathcal{A}$, take $(B_i) \subset \mathcal{B}$ so that $A_i = f^{-1}(B_i)$ and observe that

$$\begin{aligned}\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} f^{-1}(B_i) &= \{x \in X \mid \exists i \geq 1 \text{ such that } f(x) \in B_i\} \\ &= \left\{x \in X \;\middle|\; f(x) \in \bigcup_{i=1}^{\infty} B_i\right\} \\ &= f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) \in \mathcal{A}\end{aligned}$$

because $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}$. This proves that $\mathcal{A}$ is a σ-algebra.

(b) If $\mathcal{A}$ is a σ-algebra on X, show that

$$\{E \subseteq Y : f^{-1}(E) \in \mathcal{A}\}$$

is a σ-algebra on Y.

Solution: Denote this collection of sets by $\mathcal{B}$ and observe first that $f^{-1}(Y) = X \in \mathcal{A}$, so that $Y \in \mathcal{B}$. Moreover, again, $f^{-1}(B^c) = (f^{-1}(B))^c \in \mathcal{A}$ if $B \in \mathcal{B}$, so that $B^c \in \mathcal{B}$ too, showing closedness under complements. Finally, given a sequence $(B_i) \in \mathcal{B}$, we have as before

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) \in \mathcal{A},$$

so $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}$.

Exercise 1.5.
Let X be a set and $\{A_n\}_{n=1}^\infty$ be a collection of subsets of X.

(a) Show the following:

$$\limsup_{n\to+\infty} A_n = \Big\{x \in X \mid \forall N \geq 1, \exists n \geq N : x \in A_n\Big\}$$
$$\liminf_{n\to+\infty} A_n = \Big\{x \in X \mid \exists N \geq 1, \forall n \geq N : x \in A_n\Big\}$$

Solution: Observe that:

$$\begin{aligned}\limsup_{n\to+\infty} A_n &= \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} A_n = \Big\{x \in X \mid \forall N \geq 1 : x \in \bigcup_{n\geq N} A_n\Big\} \\ &= \Big\{x \in X \mid \forall N \geq 1, \exists n \geq N : x \in A_n\Big\}\end{aligned}$$

The other case follows similarly. □

(b) Show that $\liminf A_n \subset \limsup A_n$.

Solution: Using the characterisation of the previous exercise, this is immediate. □

(c) Assume $X = \{1, 2, \dots, 6\}^{\mathbb{N}}$ and $A_m = \{(x_n)_{n=1}^\infty \in X \mid x_m = 6\}$. Interpreting X as the possible outcomes of throwing a dice infinitely often and A_m as the subset of all outcomes where your m-th throw is a 6, give an interpretation of $\limsup A_m$ and $\liminf A_m$.

Solution: The set $\limsup A_m$ is the subset of outcomes, where at every point N in time, you will throw another 6 in a later turn. Therefore, $\limsup A_m$ contains all outcomes where you throw a 6 infinitely often. On the other hand, $\liminf A_m$ consists of all outcomes where, after a certain point in time, you exclusively throw 6. □

Exercise 1.6.
Let μ be a measure on a set X, and let $\{A_n\}_{n=1}^\infty$ be a sequence of subsets of X satisfying

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

Consider the set

$$E = \{x \in X : x \text{ belongs to } A_n \text{ for infinitely many } n\} = \limsup_{n\to+\infty} A_n,$$

show that $\mu(E) = 0$.

Solution: For every n, we define

$$E_n := \bigcup_{i=n}^{\infty} A_i.$$

It is straightforward to see that for all n, we have the inclusion $E \subset E_n$. Therefore, we obtain the following for all n

$$\mu(E) \leq \mu(E_n) \leq \sum_{i=n}^{\infty} \mu(A_i)$$

due to the subadditivity of μ. Because of

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty$$

we obtain

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(A_i) = 0,$$

which yields

$$\mu(E) = 0.$$

□