Exercise 2.1.
Prove that the system of elementary sets

$$\mathcal{A} := \{A \subset \mathbb{R}^n \mid A \text{ is the union of finitely many disjoint intervals}\}$$

is an algebra.

Solution: To prove that the collection of elementary sets $\mathcal{A}$ is an algebra, we need to show that $\mathbb{R}^n \in \mathcal{A}$ as well as the closedness of $\mathcal{A}$ with respect to taking complements and finite unions.

It is easy to see that $\mathbb{R}^n$ is an interval (see definition in Lecture Notes, $a, b = \pm\infty$ is allowed). Therefore, it belongs to $\mathcal{A}$. Let now $A = \bigcup_{k=1}^m A_k$ where A_k are disjoint intervals. The complements of the A_k can be expressed as:

$$A_k^c = \bigcup_{i=1}^{p(n)} B_{k,i}$$

where $p(n)$ depends on the dimension of $\mathbb{R}^n$ and determines how many pieces are needed to express the complement as a union of intervals. As a result, A_k^c is again in $\mathcal{A}$. Now, using de Morgan, we see:

$$A^c = \left(\bigcup_{k=1}^m A_k\right)^c = \bigcap_{k=1}^m A_k^c \ ,$$

However, it is obvious (since the intersection of two intervals is another interval), that the intersection of two sets $A, B \in \mathcal{A}$ lies again in $\mathcal{A}$. Therefore, we have shown $A^c \in \mathcal{A}$.

Finally, let $A_k = \bigcup_{l=1}^{n_k} A_{kl} \in \mathcal{A}$ where A_{kl} are pairwise disjoint intervals for $k = 1, \ldots, m$ and $l = 1, \ldots, n_k$. In this case, $\bigcup_{k=1}^m A_k = \bigcup_{k=1}^m \bigcup_{l=1}^{n_k} A_{kl}$ is a finite union of intervals. In addition, they can be chosen to be disjoint. To see this, let us consider the case $m = 2$, the general case follows by repeated application of the case $m = 2$. For all $l \in \{1, \ldots, n_2\}$, we define:

$$\tilde{A}_{2l} := A_{2l} \setminus \bigcup_{j=1}^{n_1} A_{1j} = A_{2l} \cap \bigcap_{j=1}^{n_1} A_{1j}^c.$$

As we argued before, A_{1j}^c is again an elementary subset and the finite intersection of elementary subsets is again elementary (consider their decomposition into disjoint cubes to see this). Therefore, $\tilde{A}_{2l}$ is elementary. Moreover, observe that all $\tilde{A}_{2l}$ are pairwise disjoint with each other and each of the A_{1j}. Therefore, using their decomposition into disjoint cubes, we can deduce that:

$$A_1 \cup A_2 \in \mathcal{A}.$$

Consequently, we see $\bigcup_{k=1}^m A_k \in \mathcal{A}$. This yields that $\mathcal{A}$ is an algebra. □

A more direct proof can be given as follows: again, we are given a finite collection of elementary sets $A_1, \ldots, A_m$. For each $1 \leq k \leq n$, let

$$-\infty =: a_0^k < a_1^k < a_2^k < \ldots < a_{q_k-1}^k < a_{q_k}^k := +\infty$$

be the finite collection of numbers which appear as one of the endpoints of the k-th factor of one of the intervals that constitute one of the A_j.

Namely, for each $k \in \{1, \dots, n\}$, let S^k be the union of the sets of endpoints of all the intervals I_k that are the k-th factor of one of the $I \subset A_j$, together with $\pm\infty$. Since S^k is finite, we can write it as $\{a_0^k, \dots, a_{q_k}^k\}$, where the elements are ordered increasingly and $a_0^k = -\infty, a_{q_k}^k = +\infty$.

Consider then the finite collection of intervals

$$\mathcal{J} = \left\{ J_1 \times \cdots \times J_n \,\middle|\, \text{for each } k, J_k = \{a_i^k\} \text{ with } 0 < i < q_k \text{ or } J_k = (a_i^k, a_{i+1}^k) \text{ with } 0 \leq i < q_k \right\}.$$

It is clear that the intervals in $\mathcal{J}$ partition $\mathbb{R}^n$, and that each A_j is a union of intervals in $\mathcal{J}$. Therefore the union $A_1 \cup \cdots \cup A_m$ can be written as the (disjoint) union of those $J \in \mathcal{J}$ which are contained in some A_j, and in particular is a disjoint union of intervals.

Note also that this also shows at once that $\mathcal{A}$ is closed under complements: defining $\mathcal{J}$ as above only for the elementary set A, we see that A is the union of some intervals from $\mathcal{J}$, therefore A^c is the union of the remaining ones.

Exercise 2.2.

Let (X, Σ, μ) be a measure space. A subset $A \in \Sigma$ is called μ-*atom*, if it holds $\mu(A) > 0$ and for every $B \in \Sigma$ such that $B \subset A$, we have either $\mu(A \setminus B) = 0$ or $\mu(B) = 0$.

(a) Let A be a μ-atom and $B \in \Sigma$ such that $B \subset A$. Prove that either $\mu(B) = \mu(A)$ or $\mu(B) = 0$.

Solution: If $\mu(B) = 0$, we are done. Therefore, assume $\mu(A \setminus B) = 0$. This gives by additivity and monotonicity:

$$\mu(B) \leq \mu(A) = \mu(B) + \mu(A \setminus B) = \mu(B),$$

thus yielding $\mu(A) = \mu(B)$. □

(b) Let $A \in \Sigma$ and assume $0 < \mu(A) < \infty$. Moreover, assume that for all $B \in \Sigma$ with $B \subset A$, it holds that either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Show that A is a μ-atom.

Solution: If $\mu(B) = 0$, we are done. Otherwise, we have $\mu(B) = \mu(A) < \infty$ and we observe that, by additivity, it holds:

$$\mu(B) = \mu(A) = \mu(B) + \mu(A \setminus B).$$

Because $\mu(B)$ is finite, this implies $\mu(A \setminus B) = 0$ and thus A is a μ-atom. □

(c) Assume that μ is σ-finite, that is, there is a countable collection $\{S_j\} \subset \Sigma$ with $\mu(S_j) < \infty$ and $X = \bigcup_j S_j$. Show that for every μ-atom A, it holds $\mu(A) < \infty$.

Solution: Since μ is σ-finite, we can write $X = \bigcup_{j=1}^{\infty} S_j$ with $S_j \in \Sigma$ and $\mu(S_j) < \infty$. Up to subtracting $\bigcup_{i=1}^{j-1} S_i$ from S_j for each j, we may suppose that all the S_j are pairwise disjoint and still in Σ. Define $A_j := S_j \cap A \in \Sigma$. Because $A_j \subset A$, it follows by the first part of the exercise that

$$\mu(A_j) = 0 \quad \text{or} \quad \mu(A) = \mu(A_j) \leq \mu(S_j) < \infty$$

for all $j \in \mathbb{N}$. If the second case occurs for some $j \in \mathbb{N}$, we are done. So it remains to consider the case $\mu(A_j) = 0$ for all j. Due to σ-additivity, this case implies $\mu(A) = 0$, which concludes the proof. □

Exercise 2.3.

Let X be an uncountable set and

$$\mathcal{B} := \{E \subset X \mid E \text{ or } E^c \text{ countable}\}.$$

Show that $\mu : \mathcal{B} \to [0, \infty]$ defined by

$$\mu(E) := \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{else} \end{cases}$$

is a pre-measure on $\mathcal{B}$.

Solution: Clearly $\mu(\varnothing) = 0$. Now assume that $A, \{A_j\}_j$ is a countable subfamily of $\mathcal{B}$ such that $A = \bigcup_j A_j$ and such that the A_j's are pairwise disjoint.

If A is countable, this implies that $A_j \subset A$ is also countable for every $j \in \mathbb{N}$. Therefore:

$$\mu(A) = 0 = \sum_{j=1}^{\infty} 0 = \sum_{j=1}^{\infty} \mu(A_j).$$

Otherwise, A is uncountable, which implies that there exists $j_0 \in \mathbb{N}$ such that A_{j_0} is uncountable as well. Furthermore, this means that $A_{j_0}^c$ is countable by definition of $\mathcal{B}$. Because A_j is contained in $A_{j_0}^c$ for all $j \neq j_0$ by disjointness, this shows that A_j is countable if $j \neq j_0$. Thus, we obtain

$$\mu(A) = 1 = \mu(A_{j_0}) = \sum_{j=1}^{\infty} \mu(A_j).$$

This shows that μ is a pre-measure. □

Exercise 2.4.

Let X be a set and $\mu : \mathcal{P}(X) \to [0, \infty]$ a measure on X. Denote by $\mathcal{A}_\mu$ the σ-algebra of μ-measurable subsets of X. Let $B \subset X$ be an arbitrary subset.

(a) Denote by $\mu \llcorner B$ the restriction of μ to B defined by:

$$\forall A \subset X : \quad \mu \llcorner B(A) := \mu(A \cap B).$$

Show that $\mu \llcorner B$ is a measure.

Solution: Let us define $\tilde{\mu} := \mu \llcorner B$. It is obvious that $\tilde{\mu}(\varnothing) = 0$. Moreover, let $A, \{A_j\}_{j \in \mathbb{N}}$ be a collection of subsets of X such that

$$A \subset \bigcup_{j=1}^{\infty} A_j.$$

It is clear that $A \cap B, \{A_j \cap B\}_{j \in \mathbb{N}}$ satisfy the same inclusion. Therefore, as μ is a measure:

$$\begin{aligned} \sum_{j=1}^{\infty} \tilde{\mu}(A_j) &= \sum_{j=1}^{\infty} \mu(A_j \cap B) \\ &\geq \mu(A \cap B) = \tilde{\mu}(A), \end{aligned}$$

which implies that $\tilde{\mu}$ is a measure. □

(b) Show that $\mathcal{A}_\mu$ is a subset of the σ-algebra of $(\mu \llcorner B)$-measurable sets.

Solution: Let $A \in \mathcal{A}_\mu$ and $C \subset X$ be arbitrary and notice by A being μ-measurable:

$$\begin{aligned}\tilde{\mu}(C \cap A) + \tilde{\mu}(C \setminus A) &= \mu(C \cap B \cap A) + \mu((C \setminus A) \cap B) = \mu(C \cap B \cap A) + \mu((C \cap B) \setminus A) \\ &= \mu(C \cap B) = \tilde{\mu}(C),\end{aligned}$$

implying that A is $\tilde{\mu}$-measurable. □

Exercise 2.5.

Let X, Y be two sets, $\mu : \mathcal{P}(X) \to [0, \infty]$ a measure on X and $f : X \to Y$ a map. How can we naturally define a "pushforward measure" $f_*\mu$ on Y? Prove that for such a measure, if $\mathcal{A}_\mu$ denotes the σ-algebra of μ-measurable sets in X, then the collection of sets[1]

$$f_*(\mathcal{A}_\mu) := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{A}_\mu\}$$

is a subset of the σ-algebra of $f_*\mu$-measurable subsets of Y.

Solution: Define, for $B \subseteq Y$, $f_*\mu(B) := \mu(f^{-1}(B))$. Clearly $f_*\mu(\varnothing) = \mu(f^{-1}(\varnothing)) = \mu(\varnothing) = 0$, and given a countable collection $\{B_j\} \subset \mathcal{P}(Y)$ and $B \subseteq \bigcup_j B_j$, it holds that $f^{-1}(B) \subseteq f^{-1}\left(\bigcup_j B_j\right) = \bigcup_j f^{-1}(B_j)$, so that

$$f_*\mu(B) = \mu(f^{-1}(B)) \leq \sum_j \mu\left(f^{-1}(B_j)\right) = \sum_j f_*\mu(B_j),$$

proving that $f_*\mu$ is indeed a measure on Y.

Moreover, given $B \in f_*(\mathcal{A}_\mu)$ and an arbitrary $E \subseteq Y$, it holds that $f^{-1}(E \cap B) = f^{-1}(E) \cap f^{-1}(B)$ and $f^{-1}(E \setminus B) = f^{-1}(E) \setminus f^{-1}(B)$, so that

$$\begin{aligned}f_*\mu(E) = \mu(f^{-1}(E)) &= \mu(f^{-1}(E) \cap f^{-1}(B)) + \mu(f^{-1}(E) \setminus f^{-1}(B)) \\ &= \mu(f^{-1}(E \cap B)) + \mu(f^{-1}(E \setminus B)) \\ &= f_*\mu(E \cap B) + f_*\mu(E \setminus B)\end{aligned}$$

since $f^{-1}(B)$ is μ-measurable. It follows that B is $f_*\mu$-measurable. □

[1] This is the σ-algebra introduced in Exercise 1.4 from Sheet 1.