Exercise 3.1.
Let μ be a measure on X and $A \subset X$ such that $\mu(A) < \infty$. Let $\{A_j\}_{j\in\mathbb{N}}$ be a countable family of μ-measurable subsets in X such that $A_j \subset A$ for all j. Assume that $\mu(A_j) \geq c_0 > 0$ for all $j \in \mathbb{N}$. Show that:

$$\mu\Big(\limsup_{j\to\infty} A_j\Big) \geq c_0.$$

Solution: Observe that $\tilde{A}_j := \cup_{l\geq j} A_l$ is a decreasing sequence of μ-measurable sets, such that $A_j \subset \tilde{A}_j \subset A$. Therefore, we know

$$\mu(\tilde{A}_1) \leq \mu(A) < \infty,$$

and consequently, by Theorem 1.2.13 (iii) of the Lecture Notes, we conclude:

$$\mu\Big(\limsup_{j\to\infty} A_j\Big) = \mu\Big(\bigcap_{j=1}^{\infty} \tilde{A}_j\Big) = \lim_{j\to\infty} \mu(\tilde{A}_j) \geq \liminf_{j\to\infty} \mu(A_j) = c_0.$$ □

Exercise 3.2.
Denote by λ the Lebesgue measure on $\mathbb{R}$. Let $E \subset [0,1]$ be a Lebesgue measurable set of strictly positive measure, i.e. $\lambda(E) > 0$. Show that for any $0 \leq \delta \leq \lambda(E)$, there exists a measurable subset of E having measure exactly δ.

Hint: Introduce the function which associates to $t \in [0,1]$ the measure of $[0,t] \cap E$. Is it continuous?

Solution: Consider the following function $f : [0,1] \to \mathbb{R}$:

$$f(t) = \lambda([0,t] \cap E), \ \ t \in [0,1].$$

Notice that $f(0) = 0$ as well as $f(1) = \lambda(E)$. We want to show that f is continuous. Therefore, take $0 \leq s < t \leq 1$. Due to the additivity on the disjoint subsets $[0,s] \cap E$ and $(s,t] \cap E$, it holds:

$$f(t) = \lambda([0,t] \cap E) = \lambda([0,s] \cap E) + \lambda((s,t] \cap E) \leq f(s) + t - s$$

where the monotonicity of λ was used in the last inequality. Consequently, we conclude

$$|f(t) - f(s)| \leq |t - s|,$$

which implies continuity.

The Intermediate Value Theorem states that for any δ between 0 and $\lambda(E)$, there exists a point x such that $f(x) = \delta$. As a result, the set $[0,x] \cap E$ (measurable as an intersection of measurable subsets) satisfies the desired property:

$$\lambda([0,x] \cap E) = \delta.$$ □

Exercise 3.3.
Let
$$\mathcal{A} = \{A \subset \mathbb{R}^n \mid A \text{ is the union of finitely many disjoint intervals}\}$$
denote the algebra of elementary sets in $\mathbb{R}^n$. Show that the volume function vol introduced in the lecture[1] for elementary sets is a pre-measure.

Remark: For $I = I_1 \times \ldots \times I_n$ an interval in $\mathbb{R}^n$, its volume is defined by
$$\mathrm{vol}(I) = \prod_{k=1}^{n} \mathrm{vol}(I_k),$$
where for an interval $I_k \subseteq \mathbb{R}$, $\mathrm{vol}(I_k)$ is the length of I_k.

Solution: Let $\{A_k\}_{k\in\mathbb{N}}$ be a countable, pairwise disjoint family of elementary sets and assume that $A = \bigcup_{k=1}^{\infty} A_k$ is another elementary set. We want to show:
$$\mathrm{vol}(A) = \sum_{k=1}^{\infty} \mathrm{vol}(A_k).$$

First of all, by considering instead of A_k its building blocks, we can assume that each A_k is an interval. The $\geq$ inequality is easy to see because
$$\mathrm{vol}(A) \geq \sum_{k=1}^{m} \mathrm{vol}(A_k)$$
holds for each m due to the monotonicity of the volume.

For the opposite inequality, let $\varepsilon > 0$ and take a compact elementary set $B \subset A$ such that $\mathrm{vol}(A) \leq \mathrm{vol}(B) + \varepsilon$ if $\mathrm{vol}(A) < \infty$ or such that $\mathrm{vol}(B) \geq \varepsilon^{-1}$ if $\mathrm{vol}(A) = \infty$. Then take open intervals U_k containing $A_k \cap B$ with $\mathrm{vol}(U_k) \leq \mathrm{vol}(A_k \cap B) + 2^{-k}\varepsilon$. All of these sets are easy to construct by slightly changing the endpoints of the intervals.

Since B is a compact set covered by the open sets U_k, we can extract a finite cover $U_{k_1}, \ldots, U_{k_m}$ of B and therefore
$$\mathrm{vol}(B) \leq \sum_{i=1}^{m} \mathrm{vol}(U_{k_i}) \leq \sum_{k=1}^{\infty} \mathrm{vol}(U_k) \leq \sum_{k=1}^{\infty} \Big(\mathrm{vol}(A_k \cap B) + 2^{-k}\varepsilon\Big) \leq \sum_{k=1}^{\infty} \mathrm{vol}(A_k) + \varepsilon.$$

Letting now $\varepsilon \to 0$, the left hand side converges to $\mathrm{vol}(A)$ and the right hand side converges to the sum $\sum_{k=1}^{\infty} \mathrm{vol}(A_k)$, thus proving the $\leq$.

Exercise 3.4.
Let X be any set with more than one element and consider the measure $\mu : \mathcal{P}(X) \to [0, +\infty]$ defined by:
$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{else} \end{cases}.$$

[1] Definition 1.3.1 in the Lecture Notes.

Give an example of a non-μ-measurable subset.

Solution: We prove that A is μ-measurable if and only if $A \in \{\emptyset, X\}$. Assume that $A \neq \emptyset, X$. In this case, there exist $x, y \in X$ such that $x \in A$ und $y \in A^c$. Consequently, we deduce:

$$\mu(X) = 1 \neq 2 = \mu(X \setminus A) + \mu(X \cap A).$$

As a result, any subset $A \neq \emptyset, X$ is not μ-measurable. Conversely, it always true that $\emptyset$ and X are μ-measurable. $\square$