Exercise 4.1.
Prove that the Lebesgue measure is invariant under translations, reflections and rotations, i.e. under all motions of the form

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi(x) = x_0 + Rx,$$

for $x_0 \in \mathbb{R}^n$ and $R \in O(n)$.

Hint: You may use the invariance of the Jordan measure, see Satz 9.3.2 in Struwe's lecture notes.

Solution: Let $I \subset \mathbb{R}^n$ be an interval. In this case, $\Phi(I)$ is a Jordan measurable set whose volume actually agrees with the volume of I. Due to the invariance of the Jordan measure μ with respect to these motions (see Satz 9.3.2 in Struwe's lecture notes), it holds that $\mathcal{L}^n(\Phi(I)) = \mu(\Phi(I)) = \mu(I) = \mathcal{L}^n(I)$.

Let now G be an open subset of $\mathbb{R}^n$ and $G = \bigcup_{k=1}^{\infty} I_k$ for some disjoint collection of intervals I_k (see Lemma 1.3.4 in the Lecture Notes). Then $\Phi(G)$ is again open (because Φ^{-1} is continuous) and $\Phi(G) = \bigcup_{k=1}^{\infty} \Phi(I_k)$ where $\Phi(I_k)$ are disjoint $\mathcal{L}^n$-measurable subsets. As above, we observe

$$\mathcal{L}^n(\Phi(G)) = \sum_{k=1}^{\infty} \mathcal{L}^n(\Phi(I_k)) = \sum_{k=1}^{\infty} \mathcal{L}^n(I_k) = \mathcal{L}^n(G).$$

For arbitrary subsets A, G of $\mathbb{R}^n$, it holds

$$A \subset G, G \text{ open} \quad \iff \quad \Phi(A) \subset \Phi(G), \Phi(G) \text{ open}$$

and consequently

$$\mathcal{L}^n(\Phi(A)) = \inf_{A \subset G, G \text{ open}} \mathcal{L}^n(\Phi(G)) = \inf_{A \subset G, G \text{ open}} \mathcal{L}^n(G) = \mathcal{L}^n(A). \qquad \square$$

Exercise 4.2.
Show that every countable subset of $\mathbb{R}$ is a Borel set and has Lebesgue measure zero.

Solution: The σ-algebra of Borel subsets $\mathcal{B}$ is the σ-algebra generated by all open subsets. Closed subsets belong to $\mathcal{B}$, because they are the complements of open subsets. As a result, since single points in $\mathbb{R}^n$ define closed subsets, it holds

$$A = \{a_1, a_2, \dots\} = \bigcup_{i \in \mathbb{N}} \{a_i\} \in \mathcal{B} \ ,$$

where A is any countable subset of $\mathbb{R}$.

A single point $\{a_i\}$ forms a set of Lebsegue measure 0 because

$$\mathcal{L}(\{a_i\}) = \lim_{k \to \infty} \mathcal{L}((a_i - \frac{1}{k}, a_i + \frac{1}{k})) = \lim_{k \to \infty} \frac{2}{k} = 0.$$

Combined with the σ-additivity (for disjoint measurable sets), it is immediate that for countable unions of point sets, we have

$$\mathcal{L}(A) = \mathcal{L}\Big(\bigcup_{i\in\mathbb{N}}\{a_i\}\Big) = \sum_{i\in\mathbb{N}} \mathcal{L}(\{a_i\}) = 0. \qquad \square$$

Exercise 4.3.
Show that the open ball $B(x,r) := \{y \in \mathbb{R}^n \mid |y-x| < r\}$ and the closed ball $\overline{B(x,r)} := \{y \in \mathbb{R}^n \mid |y-x| \le r\}$ in $\mathbb{R}^n$ are Jordan measurable with Jordan measure $c_n r^n$, for some constant $c_n > 0$ depending only on n.

Solution: We first show how the Jordan measure behaves under translations and dilations.

Claim 1: *If $A \subset \mathbb{R}^n$ bounded is Jordan measurable, then $A+x$ is Jordan measurable for all $x \in \mathbb{R}^n$ and $\mu(A+x) = \mu(A)$ (where μ is the Jordan measure, see Section 1.4 in the Lecture Notes).*

Proof. If $E \subset A$ is an elementary set, then $E + x$ is an elementary set contained in $A + x$ and $\operatorname{vol}(E+x) = \operatorname{vol}(E)$. This implies easily that $\underline{\mu}(A+x) = \underline{\mu}(A)$. Similarly one obtains $\overline{\mu}(A+x) = \overline{\mu}(A)$, which proves the claim above. $\square$

Claim 2: *If $A \subset \mathbb{R}^n$ bounded is Jordan measurable, then $tA = \{tx \mid x \in A\}$ is Jordan measurable for all $0 < t < \infty$ with $\mu(tA) = t^n\mu(A)$.*

Proof. Consider an elementary subset $E \subset A$, then tE is an elementary subset contained in tA with $\operatorname{vol}(tE) = t^n \operatorname{vol}(E)$. Similarly we can argue for elementary sets containing A, proving the claim. $\square$

Hence, it is sufficient to prove the result for $x = 0 \in \mathbb{R}^n$ and $r = 1$. In particular we will show that $\underline{\mu}(B(0,1)) = \overline{\mu}(\overline{B(0,1)})$, which proves directly that $B(0,1)$ and $\overline{B(0,1)}$ are Jordan measurable with the same measure $c_n := \mu(B(0,1)) = \mu(\overline{B(0,1)})$.

Consider the following set of intervals with side length 2^{-k}

$$\mathcal{I}_k = \{[a,b) \subset \mathbb{R}^n \mid a = 2^{-k}(a_1,\dots,a_n),\ b = 2^{-k}(a_1+1,\dots,a_n+1),\ a_i \in \mathbb{Z}\},$$

namely the standard partition of $\mathbb{R}^n$ with intervals of side length 2^{-k}. Now let $\mathcal{I}'_k = \{I \in \mathcal{I}_k \mid I \subset B(0,1)\}$ be the set of intervals in $\mathcal{I}_k$ contained in $B(0,1)$ and define $A_k := \bigcup_{I\in\mathcal{I}'_k} I \subset B(0,1)$.

Let k be large enough that $2^{-k}\sqrt{n} < 1$ and set $r_k := 1 - 2^{-k}\sqrt{n} > 0$. Given a point $x = (x_1,\dots,x_n) \in \overline{B(0,r_k)}$, consider the open cube $Q = (x_1 - 2^{-k}, x_1 + 2^{-k}) \times \cdots \times (x_n - 2^{-k}, x_n + 2^{-k})$, which is contained inside the ball $B(x, 2^{-k}\sqrt{n}) \subseteq B(0,1)$.

For each i, let $a_i = \lfloor 2^k x_i \rfloor \in \mathbb{Z}$ be the integer part of $2^k x_i$, so that $2^{-k}a_i \le x_i < 2^{-k}(a_i+1)$. Then it holds that $x_i - 2^{-k} < 2^{-k}a_i$ and $2^{-k}(a_i+1) \le x_i + 2^{-k}$, so

$$x_i \in [2^{-k}a_i, 2^{-k}(a_i+1)) \subset (x_i - 2^{-k}, x_i + 2^{-k}).$$

Thus we have the following inclusions:

$$x \in [2^{-k}a_1, 2^{-k}(a_1+1)) \times \cdots \times [2^{-k}a_n, 2^{-k}(a_n+1)) \subset Q \subset B(0,1).$$

Therefore x is contained in an interval which belongs to $\mathcal{I}'_k$, thus $x \in A_k$.

This shows that $\overline{B(0,r_k)} \subset A_k$, so $A_k \subset B(0,1) \subset \overline{B(0,1)} \subset r_k^{-1}A_k$. Thus

$$\overline{\mu}(\overline{B(0,1)}) \leq \operatorname{vol}(r_k^{-1}A_k) = r_k^{-n}\operatorname{vol}(A_k) \leq r_k^{-n}\underline{\mu}(B(0,1)).$$

Finally letting $k \to \infty$, since $r_k \to 1$, we get the inequality $\overline{\mu}(\overline{B(0,1)}) \leq \underline{\mu}(B(0,1))$, while the opposite inequality is trivial.

Remark. An alternative proof can be given by considering the n-dimensional closed ball $\overline{B^n(0,1)} \subset \mathbb{R}^n$ as the set of points that lie between the graphs of $-f$ and f, where $f : \overline{B^{n-1}(0,1)} \to \mathbb{R}$ is the function $f(x) = \sqrt{1-|x|^2}$. Since f is continuous and $\overline{B^{n-1}(0,1)}$ is compact, f is uniformly continuous, and the same is true for the extension $\bar{f} : \mathbb{R}^{n-1} \to \mathbb{R}$ of f which is zero outside of the unit ball.

Therefore given $\varepsilon > 0$ we can take k large enough that, for the dyadic decomposition $\mathcal{J}_k$ of $\mathbb{R}^{n-1}$ into intervals of side length 2^{-k} as above, it holds that

$$\sup_J \bar{f} - \inf_J \bar{f} \leq \varepsilon$$

for each interval $J \in \mathcal{J}_k$. Let $\mathcal{J}_k^- \subseteq \mathcal{J}_k^+ \subset \mathcal{J}_k$ be the finite collections of intervals defined by

$$\mathcal{J}_k^- := \{J \in \mathcal{J}_k \mid J \subseteq B^{n-1}(0,1)\}$$

and

$$\mathcal{J}_k^+ := \{J \in \mathcal{J}_k \mid J \cap \overline{B^{n-1}(0,1)} \neq \varnothing\}.$$

It is clear that every interval in $\mathcal{J}_k^+$ is contained in $[-1,1] \times \cdots \times [-1,1]$. We use these collections to define the elementary sets

$$A_k^- = \bigcup_{J \in \mathcal{J}_k^-} J \times (-\inf_J \bar{f}, \inf_J \bar{f}) \subset \mathbb{R}^n$$

and

$$A_k^+ = \bigcup_{J \in \mathcal{J}_k^+} J \times [-\sup_J \bar{f}, \sup_J \bar{f}] \subset \mathbb{R}^n.$$

It is then clear that $A_k^- \subseteq B^n(0,1) \subset \overline{B^n(0,1)} \subseteq A_k^+$ and that

$$\begin{aligned}
\operatorname{vol}(A_k^+) - \operatorname{vol}(A_k^-) &= \sum_{J \in \mathcal{J}_k^+} 2\sup_J \bar{f}\operatorname{vol}(J) - \sum_{J \in \mathcal{J}_k^-} 2\inf_J \bar{f}\operatorname{vol}(J) \\
&= \sum_{J \in \mathcal{J}_k^+} 2\left(\sup_J \bar{f} - \inf_J \bar{f}\right)\operatorname{vol}(J) \leq 2\varepsilon \sum_{J \in \mathcal{J}_k^+} \operatorname{vol}(J) \\
&= 2\varepsilon \operatorname{vol}\left(\bigcup_{J \in \mathcal{J}_k^+} J\right) \leq 2\varepsilon \operatorname{vol}\left([-1,1]^{n-1}\right) = 2^n\varepsilon.
\end{aligned}$$

Thus $\overline{\mu}(\overline{B(0,1)}) \leq \underline{\mu}(B(0,1)) + 2^n\varepsilon$ for every $\varepsilon > 0$, proving the Jordan measurability of the ball.

Exercise 4.4.

(a) Let $A \subset \mathbb{R}$ be a subset with Lebesgue measure $\mathcal{L}^1(A) > 0$. Show that there exists a subset $B \subset A$ which is **not** $\mathcal{L}^1$-measurable.

Solution: By the translation invariance of $\mathcal{L}^1$ and possibly taking a subset of A, we may assume $A \subset (0, 1)$. Now define $B_j := A \cap P_j$, where P_j is defined as in (1.5.4) of thte Lecture Notes. It was shown that if B_j is $\mathcal{L}^1$-measurable, it must have measure 0 due to $B_j \subset P_j$. Therefore, if all B_j are measurable, we obtain due to their pairwise disjointness and $\cup_j B_j = A$:

$$0 < \mathcal{L}^1(A) = \sum_{j=1}^{\infty} \mathcal{L}^1(B_j) = 0,$$

which contradicts our assumptions. □

(b) Find an example of a countable, pairwise disjoint collection $\{E_k\}_k$ of subsets in $\mathbb{R}$, such that

$$\mathcal{L}^1\Big(\bigcup_{k=1}^{\infty} E_k\Big) < \sum_{k=1}^{\infty} \mathcal{L}^1(E_k).$$

Solution: Recall the definition of P_j from the lecture (equation (1.5.4) in the Lecture Notes). This collection yields precisely the desired example. □

Exercise 4.5.

Fix some $0 < \beta < 1/3$ and define $I_1 = [0, 1]$. For every $n \geq 1$, let $I_{n+1} \subset I_n$ be the collection of intervals obtained removing from every interval in I_n its centered open subinterval of length β^n. Then define by $C_\beta = \bigcap_{n=1}^{\infty} I_n$, the *fat Cantor set* corresponding to β.

Show that:

(a) C_β is Lebesgue measurable with measure $\mathcal{L}^1(C_\beta) = 1 - \frac{\beta}{1-2\beta}$.

Solution: The set I_n is Lebesgue measurable, since it consists of 2^{n-1} intervals, and has measure $\mathcal{L}^1(I_n) = \mathcal{L}^1(I_{n-1}) - 2^{n-2}\beta^{n-1}$ for all $n \geq 2$, with $\mathcal{L}^1(I_1) = 1$. Hence

$$\mathcal{L}^1(I_n) = 1 - \sum_{k=1}^{n-1} 2^{k-1}\beta^k = 1 - \frac{1}{2}\sum_{k=1}^{n-1}(2\beta)^k = 1 - \frac{1}{2}\left(\frac{1-(2\beta)^n}{1-2\beta} - 1\right) = 1 - \frac{\beta - 2^{n-1}\beta^n}{1-2\beta}.$$

As a result $C_\beta = \bigcap_{n=1}^{\infty} I_n$ is Lebesgue measurable with measure

$$\mathcal{L}^1(C_\beta) = \lim_{n\to\infty} \mathcal{L}^1(I_n) = 1 - \frac{\beta}{1-2\beta}.$$ □

(b) C_β is not Jordan measurable. Indeed it holds $\underline{\mu}(C_\beta) = 0$ and $\overline{\mu}(C_\beta) = 1 - \frac{\beta}{1-2\beta} > 0$.

Solution: First note that C_β has empty interior, which follows from the fact that I_n consists of 2^{n-1} intervals of length $(1 - \frac{\beta}{1-2\beta})2^{-(n-1)} + \frac{\beta^n}{1-2\beta}$, which converges to 0 as $n \to \infty$. Therefore $\underline{\mu}(C_\beta) = 0$. On the other hand $\overline{\mu}(C_\beta) \geq \mathcal{L}^1(C_\beta) = 1 - \frac{\beta}{1-2\beta}$ and this is actually an equality since I_n is an elementary set for all $n \geq 1$ and therefore $\overline{\mu}(C_\beta) \leq \inf_{n\geq 1} \mathcal{L}^1(I_n) = 1 - \frac{\beta}{1-2\beta}$. Hence $\overline{\mu}(C_\beta) = 1 - \frac{\beta}{1-2\beta}$, which is greater than 0 for $0 < \beta < 1/3$. Hence C_β is not Jordan measurable. □