Exercise 5.1.
The goal of this exercise is to show that the Cantor triadic set C is uncountable. For that, recall quickly the construction of C: Every $x \in [0,1]$ can be expanded in base 3, i.e., can be written as $x = \sum_{i=1}^{\infty} d_i(x)3^{-i}$ for $d_i(x) \in \{0,1,2\}$. The set C is then defined as the set of those $x \in [0,1]$ that do not have any digit 1 in their 3-expansion, i.e.:,

$$C := \{x \in [0,1] \mid d_i(x) \in \{0,2\}, \forall i \in \mathbb{N}\}.$$

Now, the Cantor-Lebesgue function F is defined by

$$F : C \to [0,1], \quad F\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) := \sum_{i=1}^{\infty} \frac{a_i}{2^{i+1}}.$$

(a) Show that $F(0) = 0$ and $F(1) = 1$.

Solution: We see $0 = \sum_{i=1}^{\infty} 0 \cdot 3^{-i}$ and as a result, $F(0) = \sum_{i=1}^{\infty} 0 \cdot 2^{-(i+1)} = 0$. For 1 we have the expansion $1 = 0.2222...$, so $1 = \sum_{i=1}^{\infty} 2 \cdot 3^{-i}$ and therefore

$$F(1) = \sum_{i=1}^{\infty} 2 \cdot \frac{1}{2^{i+1}} = \frac{1}{2} \cdot \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1. \qquad \square$$

(b) Show that F is well-defined and continuous on C.

Solution: In general, expansions in the base 3 of an element $x \in [0,1]$ are not unique, see for example $0.1 = 0.022222.....$ However, if we restrict ourselves to expansions only using the coefficients 0 and 2, the expansion becomes unique, which shows that F is well-defined on C. (It could be easily shown that F would even be well-defined on $[0,1]$ by investigating periodic expansions more closely).

We now proceed to show that F is continuous on C. Let $\varepsilon > 0$. Take any $x \in C$ and $\{x_n\}_{n=0}^{\infty}$ any sequence in C converging to x. Take $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Because of the convergence of $\{x_n\}_{n=0}^{\infty}$ to x, there is a $M > N$ such that $|x_n - x| < 3^{-M}$, for all $n > M$. This implies that x und x_n lie in the same interval of C_n for all $n > M$, where

$$C_n = \{x \in [0,1] \mid d_i(x) \in \{0,2\}, \forall i \leq n\},$$

is the n-th approximation of the Cantor set C (see lecture). In particular, this shows that $d_i(x) = d_i(x_n)$ for any $i \leq M$. Consequently, we see that

$$|F(x_n) - F(x)| \leq \sum_{k=M+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^M} < \frac{1}{2^N} < \varepsilon,$$

which implies the continuity of F. $\square$

(c) Show that F is surjective.

Solution: Let $y \in [0,1]$ be any element. The expansion of y in the basis 2 is assumed to be $y = \sum_{k=1}^{\infty} b_k \cdot 2^{-k}$ with $b_k \in \{0,1\}$. Define $a_k := 2b_k$ for all $k \geq 1$. In this case, $x = \sum_{k=1}^{\infty} a_k \cdot 3^{-k}$

is by definition an element of C (because $a_k \in \{0,2\}$) and it holds

$$F(x) = F\left(\sum_{k=1}^{\infty} \frac{a_k}{3^k}\right) = \sum_{k=1}^{\infty} \frac{a_k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = y.$$

Therefore, F is surjective. □

(d) Conclude that C is uncountable.

Solution: F is a continuous map, which sends C surjectively onto $[0,1]$. As $[0,1]$ is uncountable, the set C has to be uncountable as well. □

Exercise 5.2.
Let E be the collection of all numbers in $[0,1]$ whose decimal expansion with respect to the basis 10 has no sevens appearing.

Recall that some decimals have two possible expansions. We are taking the convention that no expansion should be identically zero from some digit onward; for example $\frac{27}{100}$ should be written as $0,269999....$ and not as $0,27$.

Prove that E is a Lebesgue measurable set and determine its Lebesgue measure.

Solution: We can express any $x \in [0,1]$ in the following form:

$$x \;=\; 0.a_1a_2a_3\ldots$$

where $a_j \in \{0,1,\ldots,9\}$ for any $j \in \mathbb{N}$. If $x \in [0,1] \setminus E$, then at least one of the a_j is equal to 7. Therefore, for $x \in [0,1] \setminus E$, let us define

$$n(x) := \min\big\{j \in \mathbb{N} \mid a_j = 7\big\},$$

for which it holds

$$0.a_1\ldots a_{n(x)-1}7 < x \le 0.a_1\ldots a_{n(x)-1}8,$$

due to the convention that the decimal expansion does not vanish after finitely many digits and where we write $0.a_1\ldots a_{n(x)-1}8$ meaning $0.a_1\ldots a_{n(x)-1}799999\ldots$.

Therefore, we obtain the inclusion

$$[0,1] \setminus E \subset \bigcup_{n=1}^{\infty} \big\{(0.a_1\ldots a_{n-1}7, 0.a_1\ldots a_{n-1}8] \mid a_1,\ldots,a_{n-1} \neq 7\big\}$$

The converse inclusion is obvious from the definition of E. Hence $[0,1]\setminus E$ is Borel, being a countable union of pairwise disjoint intervals, which implies that E is Borel as well.

To conclude, we compute the Lebesgue measure of $[0,1] \setminus E$. For this, we first observe

$$\mathcal{L}^1\big((0.a_1\ldots a_{n-1}7, 0.a_1\ldots a_{n-1}8]\big) = \frac{1}{10^n},$$

and, noting that we have 9^{n-1} possibilities to choose the parameters $a_1, \ldots, a_{n-1}$, we find

$$\begin{aligned}\mathcal{L}^1\left([0,1] \setminus E\right) &= \sum_{n=1}^{\infty} \sum_{a_j \neq 7} \mathcal{L}^1\big((0.a_1 \ldots a_{n-1}7, 0.a_1 \ldots a_{n-1}8]\big) \\ &= \sum_{n=1}^{\infty} 9^{n-1} \cdot \frac{1}{10^n} = \frac{1}{10} \sum_{k=0}^{\infty} \left(\frac{9}{10}\right)^k = 1,\end{aligned}$$

which implies $\mathcal{L}^1(E) = 0$. □

Exercise 5.3.
Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz with constant L. Let $A \subset \mathbb{R}^n$ and $0 \leq s < +\infty$. Show that

$$\mathcal{H}^s(f(A)) \leq L^s \mathcal{H}^s(A).$$

Solution: Let $\delta > 0$ fix. Consider any covering of A of the form

$$A \subset \bigcup_{k \in \mathbb{N}} B_{r_k}(x_k), \qquad r_k < \delta.$$

Due to Lipschitz continuity, for all $x \in B_{r_k}(x_k)$ we have that $f(x) \in B_{Lr_k}(f(x_k))$, which implies

$$f(A) \subset \bigcup_{k \in \mathbb{N}} f(B_{r_k}(x_k)) \subset \bigcup_{k \in \mathbb{N}} B_{Lr_k}(f(x_k)).$$

As a result, we obtain

$$\begin{aligned}\mathcal{H}^s_\delta(A) &= \inf\Big\{\sum_{k=1}^{\infty} r_k^s \mid A \subset \bigcup_{k \in \mathbb{N}} B_{r_k}(x_k),\ r_k < \delta\Big\} \\ &\geq \inf\Big\{\sum_{k=1}^{\infty} r_k^s \mid f(A) \subset \bigcup_{k \in \mathbb{N}} B_{Lr_k}(f(x_k)),\ r_k < \delta\Big\} \\ &= \frac{1}{L^s} \inf\Big\{\sum_{k=1}^{\infty} (Lr_k)^s \mid f(A) \subset \bigcup_{k \in \mathbb{N}} B_{Lr_k}(f(x_k)),\ Lr_k < L\delta\Big\} = \frac{1}{L^s}\mathcal{H}^s_{L\delta}(f(A)),\end{aligned}$$

and by letting $\delta \to 0$, we conclude the desired inequality. □

Exercise 5.4.
Let C denote the Cantor set as defined in the lecture. Show that it holds

$$\dim_{\mathcal{H}}(C) = \frac{\ln(2)}{\ln(3)} =: s,$$

and that $2^{-s-1} \leq \mathcal{H}^s(C) \leq 2^{-s}$.

Solution: Observe that $\dim_{\mathcal{H}}(C) = s$ follows immediately from the inequalities $2^{-s-1} \leq \mathcal{H}^s(C) \leq 2^{-s}$. Consequently, it suffices to estimate the $\mathcal{H}^s$-measure of C.

By construction, we know

$$C = \bigcap_{k \in \mathbb{N}} \bigcup_{l=1}^{2^k} I_l^{(k)},$$

where $I_l^{(k)}$ are closed intervals of length 3^{-k} obtained by repeatedly removing the middle third of each emerging subinterval. We thus cover C by 2^k intervals slightly larger than the $I_l^{(k)}$, namely of radius $\frac{\lambda}{2} \cdot 3^{-k}$ with $2 > \lambda > 1$, but having the same midpoint. This enables us to directly estimate the Hausdorff measure by means of this covering:

$$\mathcal{H}^s_{2 \cdot 3^{-k}}(C) \leq \sum_{l=1}^{2^k} \left(\frac{\lambda}{2} \cdot 3^{-k} \right)^s = 2^k \lambda^s 2^{-s} 3^{-ks} = 2^k \lambda^s 2^{-s} 2^{-k} = 2^{-s} \lambda^s,$$

where we used $3^s = 2$. Observe that by letting $\lambda \to 1$, we obtain $\mathcal{H}^s_{2 \cdot 3^{-k}}(C) \leq 2^{-s}$. As we can let k go to ∞, this immediately yields $\mathcal{H}^s(C) \leq 2^{-s}$.

Next, we want to show the other inequality. Let $\{B_{r_k}(x_k)\}_{k \in \mathbb{N}}$ be a covering of C by open balls. As C is compact, we may assume without loss of generality that the covering consists of finitely many balls $B_1 = B_{r_1}(x_1), \ldots, B_N = B_{r_N}(x_N)$. For each $j = 1, \ldots, N$, there exists a $k \in \mathbb{N}$ such that

$$3^{-k-1} \leq 2r_j \leq 3^{-k},$$

which implies that B_j intersects at most one interval $I_l^{(k)}$ for $l = 1, \ldots, 2^k$. For any $m \geq k$, B_j intersects at most 2^{m-k} intervals of the form $I_l^{(m)}$ by direct considerations. Observe

$$2^{m-k} = 2^m \cdot 3^{-sk} = 2^m \cdot 3^s \cdot 3^{-s(k+1)} \leq 2^m \cdot 3^s \cdot (2r_j)^s, \tag{1}$$

by choice of k and s. Since we have a covering involving only finitely many balls, there is a m large enough, such that $3^{-m-1} \leq 2r_j$ for all $j = 1, \ldots, N$. Summing over all j the estimate (1) and observing that the number of intervals intersected by the balls must precisely be 2^m as any interval will be intersected by at least one ball, we obtain

$$2^m \leq \sum_{j=1}^{N} 2^m 3^s (2r_j)^s = 2^m 3^s 2^s \sum_{j=1}^{N} r_j^s.$$

Rearranging and cancelling yields

$$2^{-s-1} = 2^{-s} 3^{-s} \leq \sum_{j=1}^{N} r_j^s,$$

which is precisely the desired inequality due to the fact that the covering was arbitrary and the definition of $\mathcal{H}^s$. $\square$