Exercise 6.1.
For $s \geq 0$ and $\emptyset \neq A \subset \mathbb{R}^n$, we define

$$\mathcal{H}^s_\infty(A) := \inf\left\{ \sum_{k\in I} r_k^s \mid A \subset \bigcup_{k\in I} B(x_k, r_k),\ r_k > 0 \right\},$$

where the set of indices I is at most countable. One can check that $\mathcal{H}^s_\infty$ is a measure. Prove that $\mathcal{H}^{1/2}_\infty$ is not Borel on $\mathbb{R}$.

Remark. Note that the definition of $\mathcal{H}^s_\infty$ coincides with Definition 1.8.1 in the Lecture Notes for $\delta = \infty$.

Solution: We show that the interval $[0,1]$ is not $\mathcal{H}^{1/2}_\infty$-measurable, from which follows that $\mathcal{H}^{1/2}_\infty$ is not Borel on $\mathbb{R}$.

First let us prove that $\mathcal{H}^{1/2}_\infty([a,b]) = (\frac{b-a}{2})^{1/2}$ for all $a < b$. Note that the interval $B(\frac{a+b}{2}, \frac{b-a}{2} + \varepsilon)$ covers $[a,b]$ for all $\varepsilon > 0$. Therefore we have that $\mathcal{H}^{1/2}_\infty([a,b]) \leq \left(\frac{b-a}{2} + \varepsilon\right)^{1/2}$, which implies that $\mathcal{H}^{1/2}_\infty([a,b]) \leq \left(\frac{b-a}{2}\right)^{1/2}$ for arbitrariness of ε. On the other hand, given any finite or countable cover $\{B(x_k, r_k)\}_{k\in I}$ of $[a,b]$, the total length of the intervals of the covering should be at least $b-a$, namely $\sum_{k\in I} 2r_k \geq b-a$. Hence, using that $(\sum_{k\in I} r_k^{1/2})^2 \geq \sum_{k\in I} r_k$, we get

$$\sum_{k\in I} r_k^{1/2} \geq \left(\sum_{k\in I} r_k\right)^{1/2} \geq \left(\frac{b-a}{2}\right)^{1/2}.$$

Therefore we obtain that $\mathcal{H}^{1/2}_\infty([a,b]) = (\frac{b-a}{2})^{1/2}$ for all $a < b$. The same proof (with the same result) works for half-closed and open intervals.

As a result, we get

$$\mathcal{H}^{1/2}_\infty([0,2]) = 1 \neq 2^{3/2} = \mathcal{H}^{1/2}_\infty([0,1]) + \mathcal{H}^{1/2}_\infty((1,2]),$$

which proves that $[0,1]$ is not $\mathcal{H}^{1/2}_\infty$-measurable. $\square$

Exercise 6.2.
Prove the following claims.

(a) The Lebesgue measure $\mathcal{L}^n$ is a Radon measure on $\mathbb{R}^n$.

Solution: In the lecture, we have already seen that the Lebesgue measure $\mathcal{L}^n$ is Borel regular. Let $K \subset \mathbb{R}^n$ be compact. As K is bounded, for example $K \subset B_R(0)$, it obviously follows $\mathcal{L}^n(K) \leq \mathcal{L}^n(B_R(0)) = \omega_n R^n < \infty$. $\square$

(b) The Hausdorff measure $\mathcal{H}^s$ is not a Radon measure for $s < n$, but it is a Radon measure for $s \geq n$.

Solution: From the lecture, we know that $\mathcal{H}^s$ is Borel regular for all $s > 0$. It therefore suffices to check if $\mathcal{H}^s(K) < \infty$ for compact subsets $K \subset \mathbb{R}^n$. Since $\mathcal{L}^n(A) = C_n \mathcal{H}^n(A)$ for any measurable $A \subset \mathbb{R}^n$, where $C_n < \infty$ is a constant, it is immediately clear that $\mathcal{H}^n(K) < \infty$ for all compact K. By Lemma 1.8.5 in the Lecture Notes, it is obvious that for any compact $K \subset \mathbb{R}^n$ with positive Lebesgue measure, we have $\mathcal{H}^s(K) = 0$ for $s > n$ and $\mathcal{H}^s(K) = \infty$ for $s < n$. This finishes the proof. $\square$

(c) If μ is a Radon measure and $A \subset \mathbb{R}^n$ is μ-measurable, then $\mu \llcorner A$ given by

$$(\mu \llcorner A)(B) := \mu(A \cap B), \quad \forall\, B \subset \mathbb{R}^n$$

is a Radon measure as well.

Solution: We showed in Exercise 2.3 (a) that $\nu := \mu \llcorner A$ is a measure. For compact sets $K \subset \mathbb{R}^n$, it therefore holds

$$\nu(K) = \mu(A \cap K) \leq \mu(K) \leq \infty$$

because μ is a Radon measure by assumption. Moreover, by Exercise 2.3 (b), ν is a Borel measure. It remains to check whether ν is Borel regular. Let $B \subset \mathbb{R}^n$ and assume wlog $\mu(B) < \infty$ (otherwise consider $B \cap Q_l$ for a disjoint partition of $\mathbb{R}^n = \cup_{l\in\mathbb{N}} Q_l$ such that $\mu(Q_l) < \infty$). Choose C and D Borel sets such that $A \cap B \subset C$ and $B \backslash A \subset D$ as well as

$$\mu(A \cap B) = \mu(C), \qquad \mu(D) = \mu(B\backslash A) \leq \mu(B) < \infty.$$

Since $A \cap B \subset A \cap C \subset C$, we have

$$\mu(A \cap B) \leq \mu(A \cap C) \leq \mu(C) = \mu(A \cap B).$$

Therefore $\nu(C) = \mu(A \cap C) = \mu(A \cap B) = \nu(B)$. Moreover, since D is μ-measurable and $B \setminus A \subset D \setminus A$, it follows the relation

$$\nu(D) = \mu(D \cap A) = \mu(D) - \mu(D\backslash A) \leq \mu(D) - \mu(B\backslash A) = 0.$$

Finally, define the Borel set $E := C \cup D$ Borel and notice $B \subset E$ as well as

$$\nu(B) \leq \nu(E) \leq \nu(C) + \nu(D) = \nu(B),$$

which is the desired result. □

Exercise 6.3.
Given any subset $A \subset \mathbb{R}^n$, show that $\dim_{\mathcal{H}}(A) = \sup\{t \geq 0 \mid \mathcal{H}^t(A) = +\infty\}$.

Solution: By Lemma 1.8.5 in the Lecture Notes, we have that there exists $d \geq 0$ such that $\mathcal{H}^s(A) = \infty$ for all $s \in [0, d)$ and $\mathcal{H}^s(A) = 0$ for all $s \in (d, \infty)$. In particular, by Definition 1.8.8 of Hausdorff dimension, we have that $\dim_{\mathcal{H}}(A) = \inf\{s \geq 0 \mid \mathcal{H}^s(A) = 0\} = d$. On the other hand it is also clear that $\sup\{t \geq 0 \mid \mathcal{H}^t(A) = +\infty\} = d$, which implies the desired result. □

Exercise 6.4.
Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuous injective curve. We define the arc length of γ as

$$L(\gamma) := \sup\left\{\sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i)) \mid N \in \mathbb{N},\ a \leq t_0 \leq t_1 \leq \ldots \leq t_N \leq b\right\}.$$

Show that $\mathcal{H}^1(\mathrm{Im}(\gamma)) = \frac{1}{2}L(\gamma)$.

Solution: We first show that $\mathcal{H}^1(\mathrm{Im}(\gamma)) \le \frac{1}{2}L(\gamma)$. If $L(\gamma) = \infty$, we are done, otherwise take $n \in \mathbb{N}$ and take a sequence $a = t_0 \le t_1 \le \ldots \le t_{2^{n+1}} = b$. By choosing the sequence appropriately, we may assume

$$L(\gamma|_{[t_k,t_{k+1}]}) = \frac{1}{2}L(\gamma) \cdot 2^{-n}, \quad \forall k \in \{0, \ldots, 2^{n+1} - 1\}.$$

Let $\varepsilon > 0$ and define $\delta_n := \frac{1}{2}(L(\gamma) + 2\varepsilon) \cdot 2^{-n}$. Observe that

$$\gamma([t_{2k}, t_{2k+1}]) \cup \gamma([t_{2k+1}, t_{2k+2}]) \subset B_{\delta_n}(\gamma(t_{2k+1})),$$

which means that by taking all balls of radius δ_n around the t_{2k+1}, we have a covering of γ. Therefore, by definition of the Hausdorff measure

$$\mathcal{H}^1_{\delta_n}(\mathrm{Im}(\gamma)) \le \sum_{k=1}^{2^n} \delta_n = 2^n \delta_n = \frac{1}{2}L(\gamma) + \varepsilon,$$

where the summation bounds are due to us taking 2^n balls to cover γ. This proves the first inequality by letting ε go to 0.

Next, we show the converse inequality. If $\phi : [a,b] \to \mathbb{R}^n$ is a curve, we define

$$\mathrm{diam}(\mathrm{Im}(\phi)) := \sup\{d(\phi(x), \phi(y)) \mid x, y \in [a,b]\}.$$

We want to show the following inequality

$$2 \cdot \mathcal{H}^1(\mathrm{Im}(\phi)) \ge \mathrm{diam}(\mathrm{Im}(\phi)) \tag{1}$$

Before proving the inequality, let us show how to use it. Let $a = t_0 \le t_1 \le \ldots \le t_N = b$ be a sequence of points in $[a,b]$ and define $U_j := \mathrm{Im}(\gamma|_{[t_{j-1},t_j]})$ for any $j = 1, \ldots, N$. Observe that all U_j are pairwise disjoint up to single points which are $\mathcal{H}^1$-negligible. Therefore, we get that

$$\mathcal{H}^1(\mathrm{Im}(\gamma)) = \sum_{j=1}^{N} \mathcal{H}^1(U_j).$$

Using (1), this implies that

$$d(\gamma(t_{j-1}), \gamma(t_j)) \le \mathrm{diam}(\gamma|_{[t_{j-1},t_j]}) = \mathrm{diam}(U_j) \le 2 \cdot \mathcal{H}^1(U_j).$$

Now, given any $\varepsilon > 0$, we can assume that the partition $a = t_0 \le t_1 \le \ldots \le t_N = b$ is such that $L(\gamma) - \varepsilon \le \sum_{j=1}^N d(\gamma(t_{j-1}), \gamma(t_j))$. Hence, summing the previous inequality over j, we obtain

$$L(\gamma) - \varepsilon \le \sum_{j=1}^{N} d(\gamma(t_{j-1}), \gamma(t_j)) \le \sum_{j=1}^{N} 2 \cdot \mathcal{H}^1(U_j) = 2 \cdot \mathcal{H}^1(\mathrm{Im}(\gamma)).$$

Letting ε go to 0 gives the desired inequality and thus proves the result.

It remains to prove (1). Let $B_1, \ldots, B_N$ be a covering of the image of ϕ using balls of radii $r_1, \ldots, r_N$ and consider $x, y \in \mathrm{Im}(\phi)$. Because ϕ is continuous, the image is connected and there exists a finite subfamily of balls $B_{j_1}, \ldots, B_{j_k}$ such that $x \in B_{j_1}, y \in B_{j_k}$ and $B_{j_l} \cap B_{j_{l+1}} \neq \emptyset$ for all l. Therefore, we can choose points $z_1, \dots, z_{k-1}$, such that $z_l \in B_{j_l} \cap B_{j_{l+1}}$ (for brevity we denote x, y by z_0, z_k) and we therefore see that

$$d(x,y) \leq \sum_{l=0}^{k-1} d(z_l, z_{l+1}) \leq \sum_{l=0}^{k-1} \operatorname{diam}(B_{j_{l+1}}) = \sum_{l=0}^{k-1} 2r_{j_{l+1}} \leq 2 \cdot \sum_{j=1}^{N} r_j.$$

Choosing $x, y \in \operatorname{Im}(\phi)$ with distance equal to the diameter of the image and choosing appropriate coverings, we deduce from the definition of $\mathcal{H}^1$ that

$$\operatorname{diam}(\operatorname{Im}(\phi)) \leq 2 \cdot \mathcal{H}^1(\operatorname{Im}(\phi)).$$

□

Exercise 6.5.
Consider the continuous function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}.$$

(a) Show that the graph of f has infinite length as a curve, and therefore the set

$$A := \{(x, f(x)) \mid x \in [0,1]\}$$

has $\mathcal{H}^1(A) = \infty$.
Hint: use Exercise 6.4 to relate the length with the $\mathcal{H}^1$ measure.

Solution: Let $\gamma : [0,1] \to \mathbb{R}^2, x \mapsto (x, f(x))$ be the continuous injective curve that parametrizes A. Consider the sequence $x_k = \frac{2}{(2k+1)\pi} \in [0,1]$, so that $\sin \frac{1}{x_k} = \sin\left(k + \frac{1}{2}\right)\pi = (-1)^k$, and estimate the distance between two consecutive points in the curve:

$$\begin{aligned} d(\gamma(x_k), \gamma(x_{k+1})) &= d\left(\left(x_k, x_k \sin \frac{1}{x_k}\right), \left(x_{k+1}, x_{k+1} \sin \frac{1}{x_{k+1}}\right)\right) \\ &\geq \left| x_k \sin \frac{1}{x_k} - x_{k+1} \sin \frac{1}{x_{k+1}} \right| = \left| x_k (-1)^k - x_{k+1} \sin(-1)^{k+1} \right| \\ &= \left| (-1)^k (x_k + x_{k+1}) \right| = x_k + x_{k+1} > x_k. \end{aligned}$$

Therefore, for any $N > 0$ we may use the parameters $0 < x_N < x_{N-1} < \cdots < x_1 < 1$ in the supremum defining $L(\gamma)$, so that

$$L(\gamma) \geq \sum_{k=1}^{N-1} d(\gamma(x_k), \gamma(x_{k+1})) \geq \sum_{k=1}^{N-1} x_k = \sum_{k=1}^{N-1} \frac{2}{(2k+1)\pi}.$$

The right hand side behaves like the harmonic series and thus diverges as $N \to \infty$. Therefore $L(\gamma) = \infty$ and by Exercise 6.4, $\mathcal{H}^1(A) = \mathcal{H}^1(\operatorname{Im}(\gamma)) = \frac{1}{2} L(\gamma) = \infty$. □

(b) Show that $\mathcal{H}^s(A) = 0$ if $s > 1$.

Solution: Let $s > 1$ and consider $\varepsilon > 0$. The function $\gamma : [0,1] \to \mathbb{R}^2$ is smooth on $[\varepsilon, 1]$ and therefore is L_ε-Lipschitz for some constant L_ε. Thus, by exercise 5.3,

$$\mathcal{H}^s(\gamma([\varepsilon,1])) \leq L_\varepsilon^s \mathcal{H}^s([\varepsilon,1]).$$

However, since $\dim_{\mathcal{H}}([\varepsilon,1]) = 1$ (because $\mathcal{H}^1([\varepsilon,1]) \in (0,\infty)$), it follows that $\mathcal{H}^s([\varepsilon,1]) = 0$ for $s > 1$ and thus $\mathcal{H}^s(\gamma([\varepsilon,1])) = 0$.

Finally, writing $A = \gamma([0,1]) = \{(0,0)\} \cup \bigcup_{k=1}^{\infty} \gamma\left(\left[\frac{1}{k}, 1\right]\right)$ and using the subadditivity of $\mathcal{H}^s$ we get that $\mathcal{H}^s(A) = 0$. □

Remark. A more explicit solution can be given by covering A by small balls and using directly the definition of the Hausdorff measure.

(c) Conclude that $\dim_{\mathcal{H}}(A) = 1$.

Solution: The fact that $\mathcal{H}^1(A) = \infty$ implies that $\dim_{\mathcal{H}}(A) \geq 1$, and the fact that $\mathcal{H}^s(A) = 0$ for any $s > 1$ implies that $\dim_{\mathcal{H}}(A) \leq s$ for any $s > 1$. Therefore it must be $\dim_{\mathcal{H}}(A) = 1$. □