Exercise 7.1.
Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Show that the following statements are equivalent.

(i) $f^{-1}(U)$ is μ-measurable for every open set $U \subset \mathbb{R}$.

(ii) $f^{-1}(B)$ is μ-measurable for every Borel set $B \subset \mathbb{R}$.

(iii) $f^{-1}((-\infty, a))$ is μ-measurable for every $a \in \mathbb{R}$.

Solution: (i) $\Leftrightarrow$ (ii): As all open subsets $U \subset \mathbb{R}$ are Borel, it is obvious that (ii) $\Rightarrow$ (i). On the other hand, the following collection $\{B \subset \mathbb{R} \mid f^{-1}(B)\ \mu\text{-measurable}\}$ is a σ-algebra, see Exercise 1.4 (b). If this σ-algebra contains all open subsets, then it contains all Borel sets, which proves that (i) $\Rightarrow$ (ii).

(ii) $\Leftrightarrow$ (iii): Once more, it is clear that (ii) $\Rightarrow$ (iii), because the intervals $(-\infty, a)$ are Borel sets for all $a \in \mathbb{R}$. On the other hand, we know that $((-\infty, a))_{a\in\mathbb{R}}$ generates the Borel σ-algebra, which proves the other implication. □

Exercise 7.2.
Let (X, μ, Σ) be a measure space and $f, g : X \to \mathbb{R}$ two measurable functions on X. Show that the sets $\{x \mid f(x) = g(x)\}$ and $\{x \mid f(x) < g(x)\}$ are measurable.

Solution: Since f and g are measurable, then $h := f - g$ is measurable as well. As a result, we know that

$$\{x \mid f(x) = g(x)\} = h^{-1}(\{0\})$$

is measurable and the same holds for

$$\{x \mid f(x) < g(x)\} = h^{-1}((-\infty, 0)).$$ □

Exercise 7.3.
A function $g\colon \mathbb{R} \to \mathbb{R}$ is called Borel measurable, if for every open set $U \subset \mathbb{R}$ the set $g^{-1}(U)$ is a Borel set. Let (X, μ, Σ) be a measure space, let $f\colon X \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ be functions with f μ-measurable and g Borel measurable. Show that $g \circ f$ is μ-measurable.

Solution: Let $U \subset \mathbb{R}$ be an open subset. Since g is Borel measurable, it follows that $g^{-1}(U)$ is a Borel set. By assumption, $f^{-1}(B)$ is μ-measurable for all Borel sets $B \subset \mathbb{R}$, therefore, in particular, we know that $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is μ-measurable. □

Exercise 7.4.
In this exercise, we construct a set which is Lebesgue measurable, but not Borel and use the construction to give an example of a continuous $G : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $F : \mathbb{R} \to \mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

(a) Let $h : [0, 1] \to [0, 1]$ be the Cantor function, which is the unique monotonically increasing extension of the function $C \to [0, 1]$ seen in Exercise 5.1, where $C \subset [0, 1]$ is the Cantor set. Define $g : [0, 1] \to [0, 2]$ by $g(x) := h(x) + x$. Show that g is strictly monotone and a homeomorphism.

Solution: Strict monotonicity is a direct consequence of h being monotonically increasing and $x \mapsto x$ being strictly increasing. We just have to check whether g^{-1} is continuous. As $[0,1]$ is compact, the image under g of each closed subset is a compact subset of $[0,2]$, hence closed. By bijectivity, this implies that g is open and thus a homeomorphism. □

(b) Show that $\mathcal{L}^1(g(C)) = 1$.

Hint: Use the natural decomposition of $[0,1] \setminus C$ to deduce the result.

Solution: Observe that

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k},$$

where $I_{n,k}$ is the k-th interval removed in the n-th step of the construction of C and has length 3^{-n}. Hence we have

$$\mathcal{L}^1([0,2] \setminus g(C)) = \mathcal{L}^1(g([0,1] \setminus C)) = \mathcal{L}^1\left(g\left(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}\right)\right) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \mathcal{L}^1(g(I_{n,k})).$$

To conclude, notice that h is constant on each $I_{n,k}$. Therefore, we easily deduce $\mathcal{L}^1(g(I_{n,k})) = \mathcal{L}^1(I_{n,k}) = 3^{-n}$. Inserting this into the sequence of equations above, we conclude

$$\mathcal{L}^1([0,2] \setminus g(C)) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1,$$

which implies

$$1 + \mathcal{L}^1(g(C)) = \mathcal{L}^1([0,2] \setminus g(C)) + \mathcal{L}^1(g(C)) = \mathcal{L}^1([0,2]) = 2,$$

and this implies the desired result. □

(c) Use Exercise 4.4 (a) to find a non-measurable subset $E \subset g(C)$ and define $A := g^{-1}(E)$. Show that A is a Lebesgue zero set and thus Lebesgue measurable.

Solution: Observe that $A = g^{-1}(E) \subset g^{-1}(g(C)) = C$. As C is a Lebesgue zero set, so is A, and consequently A is Lebesgue measurable. □

(d) Show that A is not a Borel set.

Hint: Otherwise, the preimage of A with respect to continuous maps would necessarily be Borel and thus Lebesgue measurable as well.

Solution: Assume A were Borel. Then, due to g^{-1} being continuous by the first part of this exercise, we know

$$(g^{-1})^{-1}(A) = g(A) = g(g^{-1}(E)) = E \text{ is a Borel set.}$$

However, E is not Lebesgue measurable and hence not Borel, contradicting the conclusion. Therefore, A is not Borel. □

(e) Find appropriate F, G as outlined above such that $F \circ G$ is not Lebesgue measurable, using the sets and functions introduced in the previous subtasks.

Solution: Let us take $F = \chi_A$ and $G = g^{-1}$, where g and A are as previously introduced. Assume that $F \circ G$ is Lebesgue measurable. Note that $\{1\}$ is a closed subset and thus, $(F \circ G)^{-1}(\{1\})$ is Lebesgue measurable by assumption. Observe

$$(F \circ G)^{-1}(\{1\}) = G^{-1}\big(F^{-1}(\{1\})\big) = G^{-1}(A) = g(A) = E.$$

However, by choice of E, E is not Lebesgue measurable. Thus we arrive at the desired contradiction. $\square$

Exercise 7.5.
Let μ be a Borel measure on $\mathbb{R}$ and $f\colon [a,b] \to \mathbb{R}$ continuous μ-almost everywhere (i.e., the set of points where f is not continuous is a set of μ-measure zero). Show that f is μ-measurable.

Solution: Let $f : [a,b] \to \mathbb{R}$ be μ-almost everywhere continuous, i.e., the subset N of points of discontinuity of f has measure $\mu(N) = 0$. Consequently, N is measurable.

The restriction $g := f|_{[a,b]\setminus N}$ is continuous. Let now $\Omega \subset \mathbb{R}$ be open. Then $g^{-1}(\Omega)$ is open in $[a,b] \setminus N$, i.e., there exists $U \subset \mathbb{R}$ open such that

$$g^{-1}(\Omega) = U \cap ([a,b] \setminus N).$$

Since μ is a Borel measure, the sets U and $[a,b]$ are μ-measurable and, as a result, so is $g^{-1}(\Omega)$. We conclude that

$$f^{-1}(\Omega) = g^{-1}(\Omega) \cup (f^{-1}(\Omega) \cap N)$$

is measurable as a union of a measurable set and a set of measure zero. Therefore, f is μ-measurable. $\square$

Exercise 7.6.
Let μ be a Borel measure on $\mathbb{R}$. Show that every monotone function $f\colon [a,b] \to \mathbb{R}$ is μ-measurable.

Solution: As $f^{-1}((-\infty, c))$ is either an interval in $[a,b]$ or the empty set (when $c \notin f([a,b])$), f is μ-measurable according to Exercise 7.1. $\square$