Exercise 9.1.
In this exercise, we prove the linearity, monotonicity and well-definedness of the integral of simple functions, see Definitions 3.1.2 and 3.1.3 in the Lecture Notes. These results are essential to derive the corresponding properties of the general integral.

Remark. Throughout the exercise, we assume that all simple functions introduced are at least μ-integrable.

(a) Let f, g be two μ-measurable simple functions with values $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ (see Definition 3.1.1. in the Lecture Notes). Show that there exist μ-measurable, disjoint sets $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, such that

$$f = \sum_{n\in\mathbb{N}} a_n \chi_{A_n}, \quad g = \sum_{n\in\mathbb{N}} b_n \chi_{B_n},$$

and prove that the sets and values can be chosen in such a way that $A_n = B_n$ holds for all $n \in \mathbb{N}$.

Solution: Given f, define $A_n := f^{-1}(\{a_n\})$ which is a μ-measurable subset. Then it is obvious from the fact that $\{a_n\}_{n\in\mathbb{N}}$ is the set of values of f that

$$f = \sum_{n\in\mathbb{N}} a_n \chi_{A_n}.$$

Similarily for g with $B_n := g^{-1}(\{b_n\})$. Lastly, let us define $C_{n,m} := A_n \cap B_m$ for all $n, m \in \mathbb{N}$ and observe that there are countably many sets $C_{n,m}$ and they are all μ-measurable. Defining $c^f_{n,m} := a_n$ and $c^g_{n,m} := b_m$, we obtain the desired decompositions for f and g with the common collection of pairwise disjoint subsets $C_{n,m}$. □

(b) Assume that $f = \sum_{n\in\mathbb{N}} a_n \chi_{A_n}$, where $\{a_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{R}}$ is a sequence of values (not necessarily different from each other) and $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint, μ-measurable subsets. Prove that

$$\int f d\mu = \sum_{n\in\mathbb{N}} a_n \mu(A_n).$$

Solution: Let us denote by $\{c_m\}_{m\in\mathbb{N}}$ the values assumed by f. Observe that, due to the disjointness of the A_n's, we have

$$f^{-1}(\{c_m\}) := \bigcup_{n\in\mathbb{N}:\, a_n = c_m} A_n.$$

Therefore, by Definitions 3.1.2 and 3.1.3, we know:

$$\begin{aligned}\int f d\mu &= \sum_{m\in\mathbb{N}} c_m \mu\big(f^{-1}(\{c_m\})\big) = \sum_{m\in\mathbb{N}} c_m \mu\Big(\bigcup_{n:\, a_n = c_m} A_n\Big) = \sum_{m\in\mathbb{N}} c_m \sum_{n:\, a_n = c_m} \mu(A_n) \\ &= \sum_{m\in\mathbb{N}} \sum_{n:\, a_n = c_m} a_n \mu(A_n) = \sum_{n\in\mathbb{N}} a_n \mu(A_n),\end{aligned}$$

where we used the pairwise disjointness and the fact that every A_n can only be associated with exactly one of the c_m. □

(c) Let f, g be μ-measurable simple functions such that $f \leq g$ μ-almost everywhere. Then it holds

$$\int f d\mu \leq \int g d\mu.$$

Solution: First of all let us assume that $f \leq g$ pointwise everywhere. From the part (a), we know that we can find μ-measurable subsets C_n and sequences a_n, b_n, such that

$$f = \sum_n a_n \chi_{C_n}, \quad g = \sum_n b_n \chi_{C_n}.$$

By $f \leq g$, we know that $a_n \leq b_n$ due to the disjointness of the C_n. By part (b), this implies

$$\int f d\mu = \sum_{n\in\mathbb{N}} a_n \mu(C_n) \leq \sum_{n\in\mathbb{N}} b_n \mu(C_n) = \int g d\mu,$$

which is the desired result.

Finally note that it follows easily from parts (a) and (b) that, if $\tilde{f} = f$ μ-almost everywhere, then $\int f d\mu = \int \tilde{f} d\mu$. This proves the full result, when $f \leq g$ μ-almost everywhere. □

(d) Assume f, g are μ-summable simple functions (see Definition 3.1.8) and $a, b \in \mathbb{R}$. Show that $af + bg$ is a μ-summable simple function and

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu.$$

Solution: Let once more a_n, b_n and the sets C_n as in part (a). Then observe that

$$af + bg = \sum_{n\in\mathbb{N}} (aa_n + bb_n) \chi_{C_n},$$

which immediately implies that $af + bg$ is a simple function due to the disjointness of the sets C_n. Applying part (b) we get

$$\begin{aligned}\int (af + bg) d\mu &= \sum_{n\in\mathbb{N}} (aa_n + bb_n)\mu(C_n) = a \sum_{n\in\mathbb{N}} a_n \mu(C_n) + b \sum_{n\in\mathbb{N}} b_n \mu(C_n) \\ &= a \int f d\mu + b \int g d\mu,\end{aligned}$$

which is precisely the desired result. □

(e) Let f be a μ-integrable simple function. Prove that

$$\underline{\int} f d\mu = \overline{\int} f d\mu = \int f d\mu,$$

where the last integral is understood in the sense of integrals for simple functions, see Definitions 3.1.2 and 3.1.3 in the Lecture Notes.

Solution: From direct comparison and the fact that f is a simple function, we immediately conclude

$$\overline{\int} f\,d\mu \leq \int f\,d\mu \leq \underline{\int} f\,d\mu,$$

due to taking infima and suprema respectively. Therefore, it suffices to check

$$\underline{\int} f\,d\mu \leq \overline{\int} f\,d\mu.$$

For this, let g, h be simple functions such that $g \leq f \leq h$ μ-almost everywhere and observe that by part (c) we have

$$\int g\,d\mu \leq \int h\,d\mu.$$

Taking the supremum over all g and the infimum over all h, the desired result follows. □

Exercise 9.2.

(a) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of μ-measurable functions on a μ-measurable set $\Omega \subset \mathbb{R}^n$. Show that the series $\sum_{k=1}^{\infty} f_k(x)$ converges μ-almost everywhere, if

$$\sum_{k=1}^{\infty} \int_{\Omega} |f_k| d\mu < \infty.$$

Solution: Let us define

$$g_k := \sum_{j=1}^{k} |f_j|$$

and it obviously holds $g_k \leq g_{k+1}$ for all $k \geq 1$. Using monotone convergence of integrals, we see

$$\begin{aligned}\int_{\Omega} \sum_{j=1}^{\infty} |f_j|\, d\mu &= \int_{\Omega} \lim_{k\to\infty} g_k\, d\mu = \lim_{k\to\infty} \int_{\Omega} g_k\, d\mu = \lim_{k\to\infty} \int_{\Omega} \sum_{j=1}^{k} |f_j|\, d\mu \\ &= \lim_{k\to\infty} \sum_{j=1}^{k} \int_{\Omega} |f_j|\, d\mu = \sum_{j=1}^{\infty} \int_{\Omega} |f_j|\, d\mu.\end{aligned}$$

Since $\int_{\Omega} \sum_{j=1}^{\infty} |f_j|\, d\mu = \sum_{j=1}^{\infty} \int_{\Omega} |f_j|\, d\mu < \infty$, it holds $\sum_{j=1}^{\infty} |f_j| < \infty$ μ-almost everywhere. □

(b) Let $\{r_k\}_{k\in\mathbb{N}}$ be an ordering of $\mathbb{Q}\cap[0,1]$ and $(a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be such that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. Show that $\sum_{k=1}^{\infty} a_k|x-r_k|^{-1/2}$ is absolutely convergent for almost every $x \in [0,1]$ (with respect to the Lebesgue measure).

Solution: We apply part (a) to the functions $f_k(x) = a_k|x-r_k|^{-1/2}$ with μ equal to the Lebesgue measure. It holds

$$\begin{aligned}\int_0^1 |f_k(x)| dx &= |a_k| \int_{r_k}^{1} \frac{1}{\sqrt{x-r_k}} dx + \int_0^{r_k} \frac{1}{\sqrt{r_k - x}} dx \\ &= 2|a_k|(\sqrt{1-r_k} + \sqrt{r_k}) \leq 2\sqrt{2}|a_k|\ .\end{aligned}$$

Therefore, $\sum_{k=1}^{\infty} \int_0^1 |f_k| dx \leq 2\sqrt{2} \sum_{k=1}^{\infty} |a_k| < \infty$ by assumption and with part (a) of the exercise, the result follows. □

Exercise 9.3.
Find an example of a continuous bounded function $f: [0, \infty) \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$ and

$$\int_0^{\infty} |f(x)|^p dx = \infty \ ,$$

for all $p > 0$.

Solution: The function $f: [0, \infty) \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{\log(2+x)}$$

is continuous, bounded by $f(x) \leq \log(2)^{-1}$ and $\lim_{x\to\infty} f(x) = 0$. Since $\log(2+x) \leq p(2+x)^{\frac{1}{p}}$ for all $p > 0$, we get

$$\left|\frac{1}{\log(2+x)}\right|^p \geq \frac{1}{p^p(2+x)} \ ,$$

which is not integrable over $[0, \infty)$. □

Exercise 9.4.
Let $f, g : \Omega \to \overline{\mathbb{R}}$ be μ-summable functions and assume that

$$\int_A f d\mu \leq \int_A g d\mu$$

for all μ-measurable subsets $A \subset \Omega$. Show that $f \leq g$ μ-almost everywhere. Moreover, conclude that, if

$$\int_A f d\mu = \int_A g d\mu$$

for all μ-measurable subsets $A \subset \Omega$, then $f = g$ μ-almost everywhere.

Solution: Define $A := \{g < f\}$ and $A_n := \{g + \frac{1}{n} \leq f\}$ for all $n \in \mathbb{N}$. Notice that $\bigcup_{n\in\mathbb{N}} A_n = A$ and that A_n, A are measurable. Therefore, we find:

$$\frac{1}{n}\mu(A_n) + \int_{A_n} g d\mu = \int_{A_n} \left(g + \frac{1}{n}\right) d\mu \leq \int_{A_n} f d\mu \leq \int_{A_n} g d\mu.$$

Comparing the LHS and the RHS, we obtain $\mu(A_n) = 0$. Therefore, by continuity of the measure, we get $\mu(A) = 0$.

The second part of the exercise follows trivially from the first part. □

Exercise 9.5.
Let $f_n: \mathbb{R} \to \overline{\mathbb{R}}$ be Lebesgue measurable functions. Find examples for the following statements.

(a) $f_n \to 0$ uniformly, but not $\int |f_n| dx \to 0$.

Solution: The functions $f_n = \frac{1}{n} \cdot \chi_{[0,n]}$ are easily an example. $\square$

(b) $f_n \to 0$ pointwise and in measure, but neither $f_n \to 0$ uniformly nor $\int |f_n| dx \to 0$.

Solution: The functions $f_n = n \cdot \chi_{[\frac{1}{n}, \frac{2}{n}]}$ are an example. All properties are trivially true except for convergence in measure. For this, for all $\varepsilon > 0$, notice that

$$\mathcal{L}^1(|f_n - 0| > \varepsilon) \leq \frac{1}{n} \to 0. \qquad \square$$

(c) $f_n \to 0$ pointwise, but not in measure.

Solution: The functions $f_n = \chi_{[n,n+1]}$ are an example. They clearly not converge in measure as the limit would necessarily have to agree with the pointwise limit, since appropriate subsequences of a sequence converging in measure converge pointwise to the same limit. However it is obvious that this is not the case here. $\square$

Exercise 9.6.
Let $f : \Omega \to [0, \infty]$ be μ-measurable. Prove the following facts:

(a) If $\int_\Omega f d\mu = 0$, then $f = 0$ μ-almost everywhere.

Solution: By Exercise 9.4, since $0 \leq f$, it is enough to prove that for every μ-measurable set $A \subseteq \Omega$,

$$0 = \int_A 0 \, d\mu = \int_A f d\mu.$$

However this follows from the monotonicity of the integral:

$$0 \leq \int_A f d\mu \leq \int_\Omega f d\mu = 0.$$

$\square$

(b) If $\int_\Omega f d\mu < +\infty$, then $f < +\infty$ μ-almost everywhere.

Solution: Define for each $n \in \mathbb{N}$ the μ-measurable sets $A_n = \{x \in \Omega \mid f(x) > n\}$ and notice that $A := \{x \in \Omega \mid f(x) = +\infty\} = \bigcap_{n=1}^\infty A_n$. Set $C := \int_\Omega f d\mu < +\infty$. Then, again using the monotonicity of the integral,

$$n\mu(A_n) = \int_{A_n} n d\mu \leq \int_{A_n} f d\mu \leq \int_\Omega f d\mu = C,$$

so that $\mu(A_n) \leq C/n < +\infty$ for all n. It follows that $\mu(A) = 0$. $\square$