Exercise 10.1.
Let $f : \Omega \to \overline{\mathbb{R}}$ be μ-summable and $\Omega_1 \subseteq \Omega$ be μ-measurable. Show that $f_1 := f|_{\Omega_1}$ and $f\chi_{\Omega_1}$ are μ-summable on Ω_1 and Ω respectively, and that

$$\int_{\Omega_1} f_1 \, d\mu = \int_{\Omega} f \, \chi_{\Omega_1} d\mu.$$

Solution: Let us first show that the statement holds when f is simple: in this case, also f_1 is simple and

$$\begin{aligned}\int_{\Omega_1} f_1 \, d\mu &= \sum_{a \in \overline{\mathbb{R}}} a\mu(\{x \in \Omega_1 \mid f_1(x) = a\}) \\ &= \sum_{a \in \overline{\mathbb{R}}} a\mu(\{x \in \Omega \mid f(x) = a\} \cap \Omega_1) \\ &= \sum_{a \in \overline{\mathbb{R}}} a\mu(\{x \in \Omega \mid f(x)\chi_{\Omega_1}(x) = a\}) \\ &= \int_{\Omega} f\chi_{\Omega_1} \, d\mu.\end{aligned}$$

In the third equality we have used that, for $a \neq 0$, $f(x)\chi_{\Omega_1}(x) = a$ if and only if $f(x) = a$ and $x \in \Omega_1$.

We now take a general f and start by showing that f_1 is μ-summable: since $\int_\Omega |f| < \infty$, there exists a simple function $F : \Omega \to \overline{\mathbb{R}}$ such that $|f| \leq F$ and $\int_\Omega F < \infty$. Then the simple function $F_1 \geq f_1$ can be used to control $\int_{\Omega_1} f_1 \, d\mu \leq \int_{\Omega_1} F_1 \, d\mu = \int_\Omega F\chi_{\Omega_1} \, d\mu \leq \int_\Omega F \, d\mu < \infty$.

It is also clear that $f\chi_{\Omega_1}$ is μ-summable, since $\int_\Omega |f\chi_{\Omega_1}| \, d\mu \leq \int_\Omega |f| < \infty$. It follows that both functions are μ-integrable. To show the equality of the integrals, let g be a simple function on Ω_1 such that $g \leq f_1$ μ-a.e. in Ω_1 and h a simple function on Ω such that $h \geq f\chi_{\Omega_1}$ μ-a.e. in Ω. Then $h \geq f$ μ-a.e. in Ω_1, so that $h_1 \geq f_1 \geq g$. Moreover $h \geq 0$ μ-a.e. outside of Ω_1, so $h \geq h\chi_{\Omega_1}$. Thus,

$$\int_{\Omega_1} g \, d\mu \leq \int_{\Omega_1} h_1 \, d\mu = \int_\Omega h\chi_{\Omega_1} \, d\mu \leq \int_\Omega h \, d\mu.$$

This implies, after taking the supremum over g and the infimum over h:

$$\int_{\Omega_1} f_1 \, d\mu = \underline{\int_{\Omega_1}} f_1 \, d\mu \leq \overline{\int_\Omega} f\chi_{\Omega_1} \, d\mu = \int_\Omega f\chi_{\Omega_1} \, d\mu.$$

The proof of the converse inequality is entirely analogous. □

Exercise 10.2.
Show that if $f : \Omega \to \overline{\mathbb{R}}$ is a μ-summable function and $\Omega_1 \subseteq \Omega$ has $\mu(\Omega_1) = 0$, then

$$\int_{\Omega_1} f \, d\mu = 0.$$

Solution: Observe that the function $f\chi_{\Omega_1}$ is zero μ-almost everywhere. Therefore we can apply Exercise 10.1 and get that

$$\int_{\Omega_1} f\, d\mu = \int_{\Omega} f\chi_{\Omega_1}\, d\mu = 0. \qquad \square$$

Exercise 10.3.
By applying Lebesgue's Theorem to the counting measure on $\mathbb{N}$, show that

$$\lim_{n\to\infty} n\sum_{i=1}^{\infty} \sin\left(\frac{2^{-i}}{n}\right) = 1.$$

Solution: Let μ be the counting measure on $\mathbb{N}$. Define $f_n\colon \mathbb{N} \to \mathbb{R}$ as $f_n(i) = n\sin(2^{-i}/n)$. Note that $f_n(i) \le 2^{-i}$ and that the function $i \mapsto 2^{-i}$ is μ-summable. Hence we can apply Lebesgue's Theorem and obtain

$$\lim_{n\to\infty} n\sum_{i=1}^{\infty} \sin\left(\frac{2^{-i}}{n}\right) = \lim_{n\to\infty}\int_{\mathbb{N}} f_n d\mu = \int_{\mathbb{N}} \lim_{n\to\infty} f_n d\mu = \sum_{i=1}^{\infty} 2^{-i} = 1. \qquad \square$$

Exercise 10.4.
Let λ be the Lebesgue measure on $\mathbb{R}$ and f a nonnegative summable function on $(\mathbb{R}, \lambda)$. Show that the following equality of Lebesgue integrals holds:

$$\int_{\mathbb{R}} f d\lambda = \int_0^{+\infty} \lambda(\{f > s\}) ds.$$

Hint: In a first instance prove the equality when f is a simple function; in this case, make a picture of f and of the function $s \mapsto \lambda(\{f > s\})$ and interpret the two sides according to the definition of the Lebesgue integral.

Solution: Let f be simple and write $f = \sum_{i=1}^{\infty} a_i\chi_{A_i}$ with $\{A_i\}$ pairwise disjoint. Then, although $\lambda(\{f > s\})$ is not necessarily a simple function of s, it can be written as an infinite sum of characteristic functions with weights as follows: first note that

$$\{f > s\} = \left\{\sum_{i=1}^{\infty} a_i\chi_{A_i} > s\right\} = \bigcup_{i:a_i>s} A_i$$

and that this union is disjoint, so that

$$\lambda(\{f > s\}) = \sum_{i:a_i>s} \lambda(A_i) = \sum_{i=1}^{\infty} \lambda(A_i)\chi_{[0,a_i)}(s)$$

for $s \geq 0$. Now we use Beppo Levi's monotone convergence theorem to interchange the sum and the integral and deduce the equality:

$$\begin{aligned}\int_0^{+\infty} \lambda(\{f > s\})ds &= \int_0^{+\infty} \sum_{i=1}^{\infty} \lambda(A_i)\chi_{[0,a_i)}(s)ds \\ &= \sum_{i=1}^{\infty} \lambda(A_i) \int_0^{+\infty} \chi_{[0,a_i)}(s)ds = \sum_{i=1}^{\infty} a_i \lambda(A_i) = \int_{\mathbb{R}} f \, d\lambda.\end{aligned}$$

Next, we consider an arbitrary nonnegative summable function f. If $g \leq f$ is a simple function, we observe that $s \mapsto \lambda(\{g > s\})$ is again a simple function which is less than or equal to $s \mapsto \lambda(\{f > s\})$. Analogous observations hold for simple functions $g \geq f$.

Since f is summable, we have:

$$\int f = \overline{\int} f = \underline{\int} f,$$

and therefore, $s \mapsto \lambda(\{f > s\})$ is also summable (by directly considering simple functions as in the definition) and we have

$$\int_0^{+\infty} \lambda(\{f > s\})ds = \int f. \qquad \square$$

Remark: The function $s \mapsto \lambda(\{f > s\})$ is monotonically decreasing in s, resulting in the following inequality being trivial:

$$s \leq t \Rightarrow \{f > s\} \supset \{f > t\} \Rightarrow \lambda(\{f > s\}) \geq \lambda(\{f > t\}).$$

Consequently, the function $s \mapsto \lambda(\{f > s\})$ is Riemann integrable.

The equation proven in this exercise could be taken as the definition of the Lebesgue integral, if the integral on the right hand side is thought as a Riemann integral.

Exercise 10.5.
For all $n \in \mathbb{N}$, let $f_n \colon [0,1] \to \mathbb{R}$ be defined by:

$$f_n(x) = \frac{n\sqrt{x}}{1 + n^2x^2}.$$

Prove that:

(a) $f_n(x) \leq \frac{1}{\sqrt{x}}$ on $(0,1]$ for all $n \geq 1$.

Solution: We would like to show that $\frac{n\sqrt{x}}{1+n^2x^2} \leq \frac{1}{\sqrt{x}}$. This is equivalent to

$$nx \leq 1 + n^2x^2 \Leftrightarrow (1 - nx)^2 + nx \geq 0$$

which is true for all $x \in [0,1]$. $\square$

(b) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

Solution: Let us start with the following observation

$$\frac{n\sqrt{x}}{1+n^2x^2} \leq \frac{n\sqrt{x}}{n^2x^2} \leq \frac{1}{nx\sqrt{x}}.$$

Therefore, it holds

$$\lim_{n\to\infty} f_n(x) = 0$$

pointwise on $(0,1]$.

By (a), we know that the sequence f_n is always smaller than $g = \frac{1}{\sqrt{x}}$. Since g is Lebesgue integrable on $[0,1]$, we deduce by Lebegue's dominated convergence theorem and the pointwise convergence to 0 that:

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0. \qquad \square$$

Exercise 10.6.

Let $\Omega \subseteq \mathbb{R}^n$ be μ-measurable with $\mu(\Omega) < +\infty$ and let $\{f_j\}$ be a sequence of μ-summable $\overline{\mathbb{R}}$-valued functions such that $f_j \to f$ uniformly in Ω. Show that f is μ-summable and

$$\lim_{j\to\infty} \int_\Omega f_j \, d\mu = \int_\Omega f \, d\mu.$$

Solution: First note that, since $f = \lim_{j\to\infty} f_j = \liminf_{j\to\infty} f_j = \limsup_{j\to\infty} f_j$, f is μ-measurable. Moreover, uniform convergence implies that if j is greater than some j_0, then $|f(x) - f_j(x)| \leq 1$ for every $x \in \Omega$, thus $|f(x)| \leq |f_j(x)| + 1$ and therefore

$$\int_\Omega |f| \, d\mu \leq \int_\Omega |f_j| + 1 \, d\mu \leq \int_\Omega |f_j| \, d\mu + \mu(\Omega) < \infty$$

as f_j is μ-summable, which shows that f is also μ-summable. Finally, for $j \geq j_0$ we also have that $|f_j(x)| \leq |f(x)| + 1 =: g(x)$. Arguing as before and using again the fact that $\mu(\Omega) < \infty$, we see that g is μ-summable, so we can conclude thanks to Lebesgue's dominated convergence theorem. $\square$