Exercise 11.1.
Compute the limit

$$\lim_{n\to\infty}\int_a^{+\infty}\frac{n}{1+n^2x^2}\,dx$$

for every $a\in\mathbb{R}$.

Hint: recall that $\arctan x$ is a primitive of $\frac{1}{1+x^2}$.

Solution: Observe that $\frac{n}{1+n^2x^2}\le\frac{1}{nx^2}\le\frac{1}{x^2}$ for $x>0$. If $a>0$, then since the function $\frac{1}{x^2}$ is integrable on $(a,+\infty)$, we may apply Lebesgue's dominated convergence theorem and deduce that

$$\lim_{n\to\infty}\int_a^{+\infty}\frac{n}{1+n^2x^2}\,dx=\int_a^{+\infty}\lim_{n\to\infty}\frac{n}{1+n^2x^2}\,dx=\int_a^{+\infty}0\,dx=0.$$

For $a=0$ we can use the change of variables $y=nx$ and see that the integral is actually independent of n:

$$\int_0^{+\infty}\frac{n}{1+n^2x^2}\,dx=\int_0^{+\infty}\frac{1}{1+y^2}\,dy=\arctan y\Big|_0^{+\infty}=\arctan(+\infty)-\arctan(0)=\frac{\pi}{2}.$$

Finally for $a<0$ we get, by using the fact that the integrand is even:

$$\begin{aligned}\int_a^{+\infty}\frac{n}{1+n^2x^2}\,dx&=\int_{-\infty}^{+\infty}\frac{n}{1+n^2x^2}\,dx-\int_{-\infty}^{a}\frac{n}{1+n^2x^2}\,dx\\&=2\int_0^{+\infty}\frac{n}{1+n^2x^2}\,dx-\int_{-a}^{+\infty}\frac{n}{1+n^2x^2}\,dx.\end{aligned}$$

Thus using the two previous cases we deduce that the limit is π. □

Exercise 11.2.
Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$ be μ-measurable and $f\colon\Omega\to[0,+\infty]$ be μ-summable. For all μ-measurable subsets $A\subset\Omega$ define (see Section 3.5 in the Lecture Notes)

$$\nu(A)=\int_A f\,d\mu.$$

(a) Prove that ν is a pre-measure on the σ-algebra of μ-measurable sets, hence we can define its Carathéodory-Hahn extension $\nu\colon\mathcal{P}(\Omega)\to[0,+\infty]$.

Solution: Obviously we have that $\nu(\emptyset)=0$. Now consider a family $\{A_k\}_{k\in\mathbb{N}}$ of pairwise disjoint μ-measurable sets with $A=\bigcup_{k\in\mathbb{N}}A_k$. For all $k\in\mathbb{N}$, consider the function $f_k\colon\Omega\to[0,+\infty]$ defined as $f_k=f(\chi_{A_0}+\chi_{A_1}+\ldots+\chi_{A_k})$. Note that $f_k\le f_{k+1}$ for all $k\in\mathbb{N}$ and $f_k\xrightarrow{k\to\infty}f\chi_A$ pointwise. Hence by Beppo Levi's Theorem we get

$$\begin{aligned}\nu(A)&=\int_A f\,d\mu=\int_\Omega f\chi_A\,d\mu=\int_\Omega\lim_{k\to\infty}f_k\;d\mu=\lim_{k\to\infty}\int_\Omega f_k\,d\mu\\&=\lim_{k\to\infty}\sum_{i=0}^{k}\int_\Omega f\chi_{A_i}\,d\mu=\lim_{k\to\infty}\sum_{i=0}^{k}\int_{A_i}f\,d\mu=\sum_{k\in\mathbb{N}}\nu(A_k),\end{aligned}$$

where we used Theorem 3.1.15 and Lemma 3.1.17 of the Lecture Notes. Hence we proved that ν is a pre-measure and therefore can be extended to a measure $\nu\colon \mathcal{P}(\Omega) \to [0,+\infty]$. Moreover the σ-algebra Σ_ν of ν-measurable sets contains the σ-algebra Σ_μ of μ-measurable sets. □

(b) Show that ν is a Radon measure.

Solution: First note that ν is a Borel measure since μ is a Borel measure and $\Sigma_\nu \supset \Sigma_\mu$.

Now let us prove that ν is Borel regular. First consider any μ-measurable subset $A \subseteq \Omega$. Since μ is Borel regular, there exists a Borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$. We can also suppose that $\mu(B \setminus A) = 0$, for example by obtaining first $B_i \supseteq A \cap Q_i$, where $\{Q_i\}$ is the standard partition of $\mathbb{R}^n$ into unit cubes, and then setting $B = \bigcup_i B_i$. Hence it holds

$$\nu(A) = \int_A f d\mu = \int_B f d\mu - \int_{B\setminus A} f d\mu = \int_B f d\mu = \nu(B),$$

where we used that $\mu(B \setminus A) = 0$ and Corollary 3.1.18. Now let $A \subset \Omega$ be any set. By definition of Carathéodory-Hahn extension, there exist μ-measurable sets $A_k \supset A$ such that $\nu(A) = \lim_{k\to\infty} \nu(A_k)$. For what we proved just above, there exist Borel sets $B_k \supset A_k \supset A$ such that $\nu(B_k) = \nu(A_k)$. Then define the Borel set $B = \bigcap_{k\in\mathbb{N}} B_k$, for which it easily holds $\nu(B) = \nu(A)$. This proves that ν is Borel regular.

Let $K \subset \Omega$ be any compact set, then

$$\nu(K) = \int_K f d\mu < +\infty,$$

where we used that f is μ-summable. This concludes the proof that ν is a Radon measure. □

(c) Prove that ν is absolutely continuous with respect to μ.

Solution: We have already observed that $\Sigma_\nu \supset \Sigma_\mu$. Moreover, if $\mu(A) = 0$ for a subset $A \subset \Omega$, then $\nu(A) = 0$ by Corollary 3.1.18. This proves that ν is absolutely continuous with respect to μ. □

Exercise 11.3.

(a) Let $f\colon [a,+\infty) \to \mathbb{R}$ be a locally bounded function and locally Riemann integrable. Then f is $\mathcal{L}^1$-summable if and only if f is absolutely Riemann integrable in the generalized sense (namely $\mathcal{R}\int_a^\infty |f(x)|dx = \lim_{j\to\infty} \mathcal{R}\int_a^j |f(x)|dx$ exists and it is finite) and in this case

$$\int_{[a,+\infty)} f(x) d\mathcal{L}^1 = \mathcal{R}\int_a^\infty f(x)dx = \lim_{j\to+\infty} \mathcal{R}\int_a^j f(x)dx.$$

Solution: See proof of Exercise 3.6.7 (2) in the Lecture Notes.

(b) Let $f\colon [0,+\infty) \to \mathbb{R}$ be the function $f(x) = \frac{\sin x}{x}$, which is locally bounded and locally Riemann integrable. Show that f is Riemann integrable, i.e. $\mathcal{R}\int_0^\infty f(x)dx < +\infty$ but not absolutely Riemann integrable, i.e. $\mathcal{R}\int_0^\infty |f(x)|dx = \infty$. Hence f is not $\mathcal{L}^1$-summable.

Solution: In what follows we write $\int$ for the Riemann integral $\mathcal{R}\int$. We have that

$$\int_0^j \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + \int_1^j \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + \Big[-\frac{\cos x}{x}\Big]_1^j - \int_1^j \frac{\cos x}{x^2} dx.$$

Now note that $\int_0^1 \frac{\sin x}{x} dx < +\infty$, $\left[-\frac{\cos x}{x}\right]_1^j = \cos 1 - \cos j/j$ and

$$\left|\int_1^j \frac{\cos x}{x^2} dx\right| \leq \int_1^j \frac{|\cos x|}{x^2} dx \leq \int_1^j \frac{1}{x^2} dx = 1 - \frac{1}{j}.$$

Hence $\lim_{j\to\infty} \int_0^j \frac{\sin x}{x} dx$ exists and is finite. On the other hand we have that

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \geq \sum_{k\in\mathbb{N}} \int_{\pi k}^{\pi(k+1)} \left|\frac{\sin x}{x}\right| dx \geq \sum_{k\in\mathbb{N}} \frac{1}{k+1} \cdot \frac{1}{2} \cdot \frac{\pi}{3} = +\infty.$$ □

Exercise 11.4.
Construct a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n : [0,1] \to \mathbb{R}$ such that

- the f_n are Riemann integrable and $f_n \leq f_{n+1}$ (monotonically increasing sequence);
- $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to a function f which is NOT Riemann integrable (so $f_n(x) \to f(x)$ for all $x \in [0,1]$).

Check that Beppo Levi's Theorem holds for the constructed sequence.

Solution: *Example 1.* Take an enumeration $\{r_n\}_{n\in\mathbb{N}}$ of the rationals $\mathbb{Q}\cap[0,1]$. Define

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_0, r_1, ..., r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $f_n \leq f_{n+1}$ and that each f_n is Riemann integrable, since it has finitely many (precisely $n+1$) points of discontinuity. The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise (everywhere on $[0,1]$) to the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\cap[0,1] \\ 0 & \text{otherwise,} \end{cases}$$

which is not Riemann integrable.

Remark that f is indeed Lebesgue integrable and $\int_{[0,1]} f = 0 = \lim_{n\to\infty} \int_{[0,1]} f_n$, which is what Beppo Levi's Theorem states.

Example 2. Let $C = C_{1/4} = \bigcap_{n=1}^\infty I_n$ be the fat Cantor set from Exercise 10.1 with $\beta = 1/4$. This is a nowhere dense, compact subset of $[0,1]$ of 1-dimensional Lebesgue measure $\frac{1}{2}$.

Define the functions

$$f_n(x) = \begin{cases} 0 & \text{if } x \in I_n \\ 1 & \text{otherwise.} \end{cases}$$

Observe that $f_n \leq f_{n+1}$ and that each f_n is Riemann integrable, since it has finitely many (precisely $\sum_{i=1}^{n-1} 2^i$) points of discontinuity. The pointwise limit of $\{f_n\}_{n\geq 1}$ is the function

$$f(x) = \begin{cases} 0 & \text{if } x \in C \\ 1 & \text{otherwise.} \end{cases}$$

The function f is not Riemann integrable, since it is discontinuous on C, which has positive measure. If you have not seen this characterization of Riemann integrable functions, we can check the non-integrability straight from the definition. Take a point $x \in C$ such that, for any arbitrarily small neighbourhood U of it, both $U \cap C$ and $U \cap C^c$ have positive measure (if this were not possible we would contradict either the positiveness of $|C|$ or the fact that C is nowhere dense!). Then a Riemann-type step function should be 1 on U if we are approximating from above, 0 on U if we are approximating from below. So f is not Riemann integrable. Remark that:

$$\int_{[0,1]} f_n = (1 - |I_n|),$$

$$\int_{[0,1]} f = \frac{1}{2} = |C| = \lim_{n\to\infty} (1 - |I_n|) = \lim_{n\to\infty} \int_{[0,1]} f_n,$$

as expected from Beppo Levi's theorem. □

Exercise 11.5.

(a) Let μ be a Radon measure on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a μ-measurable subset. Consider a function $f\colon \Omega \times (a, b) \to \mathbb{R}$, for some interval $(a, b) \subset \mathbb{R}$, such that:

- the map $x \mapsto f(x, y)$ is μ-summable for all $y \in (a, b)$;
- the map $y \mapsto f(x, y)$ is differentiable in (a, b) for every $x \in \Omega$;
- there is a μ-summable function $g\colon \Omega \to [0, \infty]$ such that $\sup_{a<y<b} |\frac{\partial f}{\partial y}(x, y)| \le g(x)$ for all $x \in \Omega$.

Then $y \mapsto \int_\Omega f(x, y) d\mu(x)$ is differentiable in (a, b) with

$$\frac{d}{dy}\left(\int_\Omega f(x, y) d\mu(x)\right) = \int_\Omega \frac{\partial f}{\partial y}(x, y) d\mu(x)$$

for all $y \in (a, b)$.

Solution: Fix $y \in (a, b)$, let $\{h_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers converging to 0 and consider the μ-summable function

$$g_k(x) = \frac{f(x, y + h_k) - f(x, y)}{h_k}$$

for all k large enough so that $y + h_k \in (a, b)$. Note that $g_k(x) \to \frac{\partial f}{\partial y}(x, y)$ pointwise as $k \to \infty$. Moreover, by the mean value theorem, we have that

$$|g_k(x)| \le \sup_{a<y'<b} \left|\frac{\partial f}{\partial y}(x, y')\right| \le g(x).$$

We also have that $\frac{\partial f}{\partial y}(\cdot, y)$ is μ-measurable, since it is the pointwise limit of μ-measurable functions. Thus we can apply the Dominated Convergence Theorem, obtaining that $\frac{\partial f}{\partial y}(\cdot, y)$ is μ-summable and

$$\begin{aligned}\int_\Omega \frac{\partial f}{\partial y}(x, y) d\mu(x) &= \lim_{k\to\infty} \int_\Omega g_k(x) d\mu(x) = \lim_{k\to\infty} \frac{\int_\Omega f(x, y + h_k) d\mu(x) - \int_\Omega f(x, y) d\mu(x)}{h_k} \\ &= \frac{d}{dy} \int_\Omega f(x, y) d\mu(x),\end{aligned}$$

which concludes the proof. □

(b) Compute the integral

$$\phi(y) := \int_{(0,\infty)} e^{-x^2-y^2/x^2} d\mathcal{L}^1(x)$$

for all $y > 0$.

Hint: use part (a) to obtain that ϕ solves the Cauchy problem

$$\begin{cases} \phi'(y) = -2\phi(y) & \text{for } y > 0 \\ \lim_{y\to 0^+} \phi(y) = \sqrt{\pi}/2. \end{cases}$$

Solution: First note that $e^{-x^2-y^2/x^2} \leq e^{-x^2}$ is $\mathcal{L}^1$-summable for all $y > 0$ and $y \mapsto e^{-x^2-y^2/x^2}$ is differentiable in $(0, +\infty)$ for all $x > 0$. Moreover, for all $x, y > 0$, we have that

$$\left|\frac{\partial}{\partial y} e^{-x^2-y^2/x^2}\right| = \frac{2y}{x^2} e^{-x^2-y^2/x^2} \leq \frac{2e^{-x^2}}{y} \cdot \frac{y^2}{x^2} e^{-y^2/x^2} \leq \frac{2e^{-x^2}}{y} e^{-1}.$$

Hence $\frac{\partial}{\partial y} e^{-x^2-y^2/x^2}$ is controlled by the $\mathcal{L}^1$-summable function e^{-x^2}/r for all $y > r$. Therefore we can apply part (a) and obtain that

$$\begin{aligned} \phi'(y) &= -\int_{(0,\infty)} \frac{2y}{x^2} e^{-x^2-y^2/x^2} d\mathcal{L}^1(x) \overset{t=y/x}{=} -2\int_{(0,\infty)} y\frac{t^2}{y^2} e^{-t^2-y^2/t^2} \frac{y}{t^2} d\mathcal{L}^1(t) \\ &= -2\int_{(0,\infty)} e^{-t^2-y^2/t^2} d\mathcal{L}^1(t) = -2\phi(y). \end{aligned}$$

Since $\int_0^\infty e^{-x^2} d\mathcal{L}^1(x) = \sqrt{\pi}/2$, ϕ satisfies the Cauchy problem

$$\begin{cases} \phi'(y) = -2\phi(y) & \text{for } y > 0 \\ \lim_{y\to 0^+} \phi(y) = \sqrt{\pi}/2, \end{cases}$$

which has solution $\phi(y) = \sqrt{\pi} e^{-y}/2$. □

Exercise 11.6.

Let μ be a Radon measure on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$ and $f, f_k : \Omega \to \overline{\mathbb{R}}$ μ-summable functions.

(a) Show that Vitali's Theorem implies Dominated Convergence Theorem.

Solution: Let $g : \Omega \to [0, \infty]$ be μ-summable and consider $|f_k| \leq g$ and $f_k \to f$ μ-almost everywhere, where $f, f_k : \Omega \to \overline{\mathbb{R}}$ are μ-measurable functions, for $k \in \mathbb{N}$.

Since $\mu(\Omega) < +\infty$, we have the convergence $f_k \overset{\mu}{\to} f$ (see Theorem 2.4.2 in the Lecture Notes). In addition, the f_k's are uniformly μ-summable. This is due to the monotonicity of the integral for $|f_k| \leq g$ and the absolute continuity of the integral of g (see theorem below). As a result, the conditions of Vitali's theorem are satisfied and it follows that $\lim_{k\to\infty} \int_\Omega |f_k - f| d\mu = 0$. □

Let us conclude by proving the absolute continuity of the integral of g (since in the lecture we used the Dominated Convergence Theorem), namely:

Theorem. *Let $g : \Omega \to \overline{\mathbb{R}}$ be μ-summable. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all μ-measurable subsets $A \subset \Omega$ with $\mu(A) < \delta$, it holds $\int_A |g| d\mu < \varepsilon$.*

Proof. Without loss of generality, assume that $g \geq 0$ and define $g_n := \min\{g, n\}$. Then g_n converges pointwise to g μ-a.e. and by monotone convergence we have $\lim_{n\to\infty} \int_\Omega g_n d\mu = \int_\Omega g d\mu$, in particular $\lim_{n\to\infty} \int_\Omega |g - g_n| d\mu = 0$.
Now let $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ such that $\int_\Omega |g - g_N| d\mu < \varepsilon/2$. Hence, choosing $\delta = \varepsilon/(2N)$, we deduce for all measurable subsets $A \subset \Omega$ with $\mu(A) < \delta$ that

$$\int_A |g| d\mu \leq \int_A |g - g_N| d\mu + \int_A |g_N| d\mu < \int_\Omega |g - g_N| d\mu + \mu(A)N < \varepsilon.$$

This indeed proves the absolute continuity of the integral. □

(b) Let $\Omega = [0, 1]$ and $\mu = \mathcal{L}^1$. Give an example in which Vitali's Theorem can be applied but Dominated Convergence Theorem cannot, i.e., a dominating function does not exist.

Hint: look at the functions $f_n^k(x) = \frac{1}{x}\chi_{[\frac{n+k-1}{n2^{n+1}}, \frac{n+k}{n2^{n+1}})}(x)$ for $n \in \mathbb{N}$, $1 \leq k \leq n$.

Solution: For $n \in \mathbb{N}$, $1 \leq k \leq n$, consider the function $f_n^k(x) = \frac{1}{x}\chi_{[\frac{n+k-1}{n2^{n+1}}, \frac{n+k}{n2^{n+1}})}(x)$.

The sequence $\{f_n^k\}$ is uniformly μ-summable. Indeed, given $\varepsilon > 0$, choose $M \in \mathbb{N}$ with $1/M \leq \varepsilon$ and $\delta := 2^{-(M+1)}/M$, then for all $A \subset [0, 1]$ with $\mu(A) < \delta$ we have:

- if $n \geq M$ and $1 \leq k \leq n$, then

$$\int_A |f_n^k(x)| dx \leq \int_0^1 |f_n^k(x)| dx \leq 2^{n+1} \cdot \frac{1}{n2^{n+1}} = \frac{1}{n} \leq \frac{1}{M} \leq \varepsilon;$$

- if $n < M$ and $1 \leq k \leq n$, then

$$\int_A |f_n^k(x)| dx \leq 2^{n+1}\delta = \frac{2^{n+1}}{M2^{M+1}} < \frac{1}{M} \leq \varepsilon.$$

Furthermore, $f_n^k \to 0$ converges pointwise and, as a result, converges in measure. Hence (f_n^k) satisfies the conditions of Vitali's theorem. However, a dominating function would have to be larger than $1/x$, which implies non-summability over $[0, 1]$. □