Exercise 12.1.

The goal of this exercise is to compute the following Riemann integral:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{a\to\infty}\int_0^a \frac{\sin x}{x}\,dx.$$

(a) Show that the function $\Phi : (0,\infty) \to \mathbb{R}$,

$$\Phi(t) = \int_0^\infty e^{-tx}\frac{\sin x}{x}\,dx,$$

is well-defined and differentiable everywhere.

Solution: Finiteness is clear since $|\frac{\sin x}{x}| \le 1$. For the differentiability, given any sequence $h_j \to 0$, we want to apply the dominated convergence theorem to commute the integral and the limit in the following computation:

$$\begin{aligned}
\lim_{j\to\infty}\frac{\Phi(t+h_j)-\Phi(t)}{h_j} &= \lim_{j\to\infty}\int_0^\infty \frac{e^{-(t+h_j)x}-e^{-tx}}{h_j}\frac{\sin x}{x}\,dx \\
&= \int_0^\infty \lim_{j\to\infty}\frac{e^{-(t+h_j)x}-e^{-tx}}{h_j}\frac{\sin x}{x}\,dx \\
&= \int_0^\infty \frac{d}{dt}\left(e^{-tx}\right)\frac{\sin x}{x}\,dx = \int_0^\infty -e^{-tx}\sin x\,dx.
\end{aligned}$$

For that, it is enough to bound the integrands by a summable function. We can do this by using the standard estimate $|e^u - 1| \le e^{|u|}|u|$, which follows from the mean value theorem. Thus

$$\left|\frac{e^{-(t+h_j)x}-e^{-tx}}{h_j}\frac{\sin x}{x}\right| = \frac{\left|e^{-h_jx}-1\right|}{|h_j|}e^{-tx}\left|\frac{\sin x}{x}\right| \le e^{|h_j|x}xe^{-tx}\left|\frac{\sin x}{x}\right| = e^{-tx/2}|\sin x| \in L^1(0,\infty)$$

whenever $|h_j| \le t/2$, which happens for j large enough. Thus $\Phi(t)$ is differentiable with derivative

$$\Phi'(t) = \int_0^\infty -e^{-tx}\sin x\,dx. \qquad \square$$

(b) Compute $\Phi'(t)$ for $t \in (0,\infty)$.

Solution: Using the expression above we integrate twice by parts:

$$\begin{aligned}
\Phi'(t) &= -\int_0^\infty e^{-tx}\sin x\,dx \\
&= \left[e^{-tx}\cos x\right]_0^\infty - \int_0^\infty -te^{-tx}\cos x\,dx \\
&= -1 + t\int_0^\infty e^{-tx}\cos x\,dx \\
&= -1 + \left[te^{-tx}\sin x\right]_0^\infty - \int_0^\infty -t^2e^{-tx}\sin x\,dx \\
&= -1 - t^2\Phi'(t)
\end{aligned}$$

so that

$$\Phi'(t) = -\frac{1}{1+t^2}. \qquad \square$$

(c) Compute $\Phi(t)$ for $t \in (0, \infty)$.

Solution: We show first that $\Phi(t) \to 0$ as $t \to \infty$: this follows immediately from dominated convergence, since $|\frac{\sin x}{x}| \leq 1$. Therefore the fundamental theorem of calculus yields

$$\Phi(t) = -\left(\lim_{s\to\infty} \Phi(s) - \Phi(t)\right) = -\int_t^\infty \Phi'(t) = \int_t^\infty \frac{1}{1+t^2}\, dt = \frac{\pi}{2} - \arctan(t). \qquad \square$$

(d) Show that the convergence

$$\int_0^a e^{-tx}\frac{\sin x}{x}\, dx \xrightarrow{a\to\infty} \int_0^\infty e^{-tx}\frac{\sin x}{x}\, dx$$

is uniform in $t > 0$.

Hint: this part is technically more difficult. It is not true that $\int_a^\infty \left|e^{-tx}\frac{\sin x}{x}\right| dx$ converges to zero uniformly in t as $a \to \infty$. Here one has to use the cancellations of the integral, for example by seeing that

$$\sum_{k=m}^{\infty} \left| \int_{2k\pi}^{2(k+1)\pi} e^{-tx}\frac{\sin x}{x}\, dx \right|$$

converges to zero as $m \to \infty$ uniformly in t.

Solution: Given $a > 0$, let $m \in \mathbb{N}$ be such that $2\pi(m-1) < a \leq 2\pi m$ and write

$$\left| \int_a^\infty e^{-tx}\frac{\sin x}{x}\, dx \right| \leq \int_a^{2\pi m} \left| e^{-tx}\frac{\sin x}{x} \right| dx + \sum_{k=m}^{\infty} \left| \int_{2\pi k}^{2\pi(k+1)} e^{-tx}\frac{\sin x}{x} dx \right|.$$

For the first term we have:

$$\int_a^{2\pi m} \left| e^{-tx}\frac{\sin x}{x} \right| dx \leq \frac{2\pi m - a}{a} \leq \frac{2\pi}{a}.$$

On the other hand, for each term in the sum we use two changes of variables and write

$$\begin{aligned}
\int_{2\pi k}^{2\pi(k+1)} e^{-tx}\frac{\sin x}{x} &= \int_0^\pi e^{-t(2k\pi+x)}\frac{\sin(2k\pi+x)}{2k\pi+x} dx + \int_0^\pi e^{-t((2k+1)\pi+x)}\frac{\sin((2k+1)\pi+x)}{(2k+1)\pi+x} dx \\
&= \int_0^\pi \sin x \frac{e^{-t(2k\pi+x)}}{2k\pi+x} dx - \int_0^\pi \sin x \frac{e^{-t((2k+1)\pi+x)}}{(2k+1)\pi+x} dx \\
&= \int_0^\pi \sin x \left(\frac{e^{-t(2k\pi+x)}}{2k\pi+x} - \frac{e^{-t((2k+1)\pi+x)}}{(2k+1)\pi+x} \right) dx \\
&\leq \pi \left(\frac{e^{-t(2k\pi)}}{2k\pi} - \frac{e^{-t(2(k+1)\pi)}}{2(k+1)\pi} \right).
\end{aligned}$$

This is a telescoping series, so

$$\left|\int_a^\infty e^{-tx}\frac{\sin x}{x}\,dx\right| \le \frac{2\pi}{a} + \pi\sum_{k=m}^{\infty}\left(\frac{e^{-t(2k\pi)}}{2k\pi} - \frac{e^{-t(2(k+1)\pi)}}{2(k+1)\pi}\right) \le \frac{2\pi}{a} + \pi\frac{e^{-t(2m\pi)}}{2m\pi} \le \frac{2\pi}{a} + \frac{\pi}{a},$$

which tends to 0 as $a \to \infty$ uniformly in t. □

(e) Conclude that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Solution: Given $\varepsilon > 0$, uniform convergence gives us some a_0 such that for $a \ge a_0$,

$$\left|\int_0^a e^{-tx}\frac{\sin x}{x}\,dx - \int_0^\infty e^{-tx}\frac{\sin x}{x}\,dx\right| \le \frac{\varepsilon}{3} \tag{1}$$

holds for every $t > 0$. Now fix $a \ge a_0$ and choose t small enough so that $a(1 - e^{-ta}) \le \varepsilon/3$ and such that $\arctan t \le \varepsilon/3$. The first condition implies:

$$\left|\int_0^a \frac{\sin x}{x}\,dx - \int_0^a e^{-tx}\frac{\sin x}{x}\,dx\right| \le \int_0^a (1 - e^{-tx})\frac{|\sin x|}{x}\,dx \le a(1 - e^{-ta}) \le \frac{\varepsilon}{3}, \tag{2}$$

while the second condition yields:

$$\left|\frac{\pi}{2} - \int_0^\infty e^{-tx}\frac{\sin x}{x}\,dx\right| = \left|\frac{\pi}{2} - \Phi(t)\right| = \arctan t \le \frac{\varepsilon}{3}. \tag{3}$$

Finally putting together (2), (1) and (3) we see that

$$\left|\int_0^a \frac{\sin x}{x}\,dx - \frac{\pi}{2}\right| \le \varepsilon$$

for $a \ge a_0$ as we wanted to show. □

Exercise 12.2.
Let $1 \le p < \infty$. Show that if $\varphi \in L^p(\mathbb{R}^n)$ and φ is uniformly continuous, then

$$\lim_{|x|\to\infty} \varphi(x) = 0.$$

Solution: Suppose, by contradiction, that there is $\varepsilon > 0$ and a sequence $\{x_k\}$ with $|x_k| \to \infty$ and $|\varphi(x_k)| \ge \varepsilon$. Then by uniform continuity, there is $\delta > 0$ such that for every $x \in B_\delta(x_k)$ we have $|\varphi(x) - \varphi(x_k)| \le \varepsilon/2$, which implies that $|\varphi(x)| \ge \varepsilon/2$. Since $|x_k| \to \infty$, we can pass to a subsequence $\{x_{k_j}\}$ with $|x_{k_j}| > |x_{k_{j-1}}| + 2\delta$. This implies in particular that for any $j \ne j'$, $|x_{k_j} - x_{k_{j'}}| > 2\delta$, so that the balls $B_\delta(x_{k_j})$ and $B_\delta(x_{k_{j'}})$ are disjoint. Thus we get the following lower bound which shows that $\varphi \notin L^p(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |\varphi(x)|^p\,dx \ge \sum_{j=1}^{\infty}\int_{B_\delta(x_{k_j})} |\varphi(x)|^p\,dx \ge \sum_{j=1}^{\infty}\int_{B_\delta(x_{k_j})} \left(\frac{\varepsilon}{2}\right)^p\,dx = +\infty$$ □

Exercise 12.3.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set.

(a) (Generalized Hölder inequality) Consider $1 \le p_1, \dots, p_k \le \infty$ such that $\frac{1}{r} = \sum_{i=1}^k \frac{1}{p_i} \le 1$. Show that, given functions $f_i \in L^{p_i}(\Omega, \mu)$ for $i = 1, \dots, k$, it holds $\prod_{i=1}^k f_i \in L^r(\Omega, \mu)$ and

$$\left\| \prod_{i=1}^k f_i \right\|_{L^r} \le \prod_{i=1}^k \|f_i\|_{L^{p_i}}.$$

Solution: We can suppose that all p_i are finite, since it is easy to deal with $p_i = \infty$ directly. We will prove the statement by induction. For $k = 1$ there is nothing to prove. For the induction step $k-1 \to k$, we know that $\frac{1}{r} - \frac{1}{p_k} = \frac{p_k - r}{p_k r} = \sum_{j=1}^{k-1} \frac{1}{p_j}$. By the induction hypothesis, we have that $\prod_{j=1}^{k-1} f_j \in L^{\frac{p_k r}{p_k - r}}(\Omega, \mu)$ together with the estimate

$$\left\| \prod_{j=1}^{k-1} f_j \right\|_{L^{\frac{p_k r}{p_k - r}}} \le \prod_{j=1}^{k-1} \|f_j\|_{L^{p_j}}.$$

Now we apply Hölder's inequality to the functions $g_1 = \prod_{j=1}^{k-1} |f_j|^r$ and $g_2 = |f_k|^r$, with exponents $\frac{p_k}{p_k - r}$ and $\frac{p_k}{r}$ respectively:

$$\begin{aligned}
\int_\Omega \left(\prod_{j=1}^k |f_k| \right)^r &\le \left(\int_\Omega \prod_{j=1}^{k-1} |f_j|^{r \frac{p_k}{p_k - r}} \right)^{\frac{p_k - r}{p_k}} \left(\int_\Omega |f_k|^{r \frac{p_k}{r}} \right)^{\frac{r}{p_k}} \\
&= \left\| \prod_{j=1}^{k-1} f_j \right\|^r_{L^{\frac{p_k r}{p_k - r}}} \|f_k\|^r_{L^{p_k}} \le \prod_{j=1}^{k-1} \|f_j\|^r_{L^{p_j}} \cdot \|f_k\|^r_{L^{p_k}}.
\end{aligned}$$

This yields $\| \prod_{i=1}^k f_i \|_{L^r} \le \prod_{i=1}^k \|f_i\|_{L^{p_i}}$, as we wanted to show. □

(b) Prove that, if $\mu(\Omega) < +\infty$, then $L^s(\Omega, \mu) \subseteq L^r(\Omega, \mu)$ for all $1 \le r < s \le +\infty$.

Solution: Fix $1 \le r < s \le +\infty$ and define $p = rs/(s - r)$, for which it holds $\frac{1}{s} + \frac{1}{p} = \frac{1}{r}$. If $\mu(\Omega) < +\infty$, then $g = 1 \in L^p(\Omega, \mu)$, hence we can apply part (a) and obtain that, for all $f \in L^r(\Omega, \mu)$, $f = f \cdot 1 \in L^r(\Omega, \mu)$, which proves the desired inclusion. □

(c) Show that the inclusion in part (b) is strict for all $1 \le r < s \le +\infty$.

Solution: For all $1 \le r < +\infty$, consider the function $f\colon (0, 1/2) \to \mathbb{R}$ given by

$$f(x) = \left(\log^2\left(\frac{1}{x}\right) x^{1/r} \right)^{-1}.$$

Note that $f \in L^r$ since

$$\begin{aligned}
\int_0^{1/2} \left(\log^2\left(\frac{1}{x}\right) x^{1/r} \right)^{-r} dx &= \lim_{\varepsilon \to 0} \int_\varepsilon^{1/2} \left(\log^{2r}\left(\frac{1}{x}\right) x \right)^{-1} \\
&= \lim_{\varepsilon \to 0} \left[\frac{1}{(2r-1) \log^{2r-1}(1/x)} \right]_\varepsilon^{1/2} = \frac{1}{(2r-1)\log^{2r-1}(2)}.
\end{aligned}$$

On the other hand $f \notin L^s$ for all $s > r$: in this case we can choose $0 < t < \frac{1}{r} - \frac{1}{s}$ and estimate $\log^2\left(\frac{1}{x}\right) \le Cx^{-t}$ with a constant $C > 0$. Then follows

$$\left(\log^2\left(\frac{1}{x}\right) x^{1/r}\right)^{-1} \ge \frac{1}{C} x^{t-\frac{1}{r}}$$

with $s\left(t - \frac{1}{r}\right) < -1$, which is not integrable. □

Exercise 12.4.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set with $\mu(\Omega) < +\infty$. Consider a function $f\colon \Omega \to \overline{\mathbb{R}}$ such that $fg \in L^1(\Omega, \mu)$ for all $g \in L^p(\Omega, \mu)$. Prove that $f \in L^q(\Omega, \mu)$ for all $q \in [1, p')$, where $p' = \frac{p}{p-1}$ is the conjugate of p.

Solution: First note that, taking $g = 1 \in L^p(\Omega, \mu)$, we get that $f \in L^1(\Omega, \mu)$. Hence we can consider the function $g = |f|^{1/p} \in L^p(\Omega, \mu)$ and we get that $|f|^{1+1/p} \in L^1(\Omega, \mu)$. Therefore we can choose $g = |f|^{1/p+1/p^2} \in L^p(\Omega, \mu)$ and get that $|f|^{1+1/p+1/p^2} \in L^1(\Omega, \mu)$.

Repeating again the same argument by induction, we get that $|f|^{p_n} \in L^1(\Omega, \mu)$ for all $n \in \mathbb{N}$, where $p_n = 1 + \frac{1}{p} + \cdots + \frac{1}{p^n} = \frac{1-1/p^{n+1}}{1-1/p}$. In particular we have that $f \in L^{p_n}(\Omega, \mu)$ for all $n \in \mathbb{N}$, which implies that $f \in L^q(\Omega, \mu)$ for all $1 \le q \le p_n$ by Exercise 12.3 (b). Now note that $p_n \to p'$ as $n \to \infty$, thus $f \in L^q(\Omega, \mu)$ for all $1 \le q < p'$, as desired. □

Exercise 12.5.
Let μ be a Radon measure on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ a μ-measurable set.

(a) Show that any $f \in \bigcap_{p\in\mathbb{N}^*} L^p(\Omega, \mu)$ with $\sup_{p\in\mathbb{N}^*} \|f\|_{L^p} < +\infty$ lies in $L^\infty(\Omega, \mu)$.
Hint. Tchebychev' inequality.

Solution: Let $C = \sup_{p\in\mathbb{N}^*}\|f\|_{L^p}$ and $\varepsilon > 0$. Using Tchebychev' inequality, we have:

$$\begin{aligned}\mu(\{|f| \ge C + \varepsilon\}) = \mu(\{|f|^p \ge (C + \varepsilon)^p\}) &\le \frac{1}{(C+\varepsilon)^p} \int_\Omega |f|^p d\mu \\ &\le \left(\frac{C}{C+\varepsilon}\right)^p \to 0\ , \quad \text{as } p \to \infty.\end{aligned}$$

Hence $\mu(\{|f| \ge C + \varepsilon) = 0$ and we deduce $f \in L^\infty$. Since $\varepsilon > 0$ was arbitrary, by

$$\mu(\{|f| > C\}) = \mu(\cup_{n\in\mathbb{N}}\{|f| \ge C + 1/n\}) \le \sum_{n\in\mathbb{N}} \mu(\{|f| \ge C + 1/n\}) = 0$$

we conclude $\|f\|_{L^\infty} \le C$. □

(b) Show that if $\mu(\Omega) < +\infty$, then for any f as in part (a) we have that $\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}$.

Solution: Choose a sequence $(p_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\|f\|_{L^{p_k}} = \liminf_{p\to\infty}\|f\|_{L^p}$ and let $\varepsilon > 0$. Take k_0, such that $\|f\|_{L^{p_k}} \le \liminf_{p\to\infty}\|f\|_{L^p} + \varepsilon$ for $k \ge k_0$. Analogous to (a), it follows $\|f\|_{L^\infty} \le \liminf_{p\to\infty}\|f\|_{L^p} + \varepsilon$ and by letting $\varepsilon \downarrow 0$, we deduce $\|f\|_{L^\infty} \le \liminf_{p\to\infty}\|f\|_{L^p}$.

For the opposite bound, choose a sequence $(p_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\|f\|_{L^{p_k}} = \limsup_{p\to\infty}\|f\|_{L^p}$. For $q > p$, we have $\|f\|_{L^q}^q \le \|f\|_{L^p}^p \|f\|_{L^\infty}^{q-p}$. Take $p > 1$ and $k_0 \in \mathbb{N}$, such that $p_k > p$ for $k \ge k_0$. It follows

$$\|f\|_{L^{p_k}} \le \|f\|_{L^p}^{\frac{p}{p_k}} \|f\|_{L^\infty}^{1-\frac{p}{p_k}} \xrightarrow{k\to\infty} 1 \cdot \|f\|_{L^\infty}.$$

As a result, we see $\limsup_{p\to\infty}\|f\|_{L^p} = \lim_{k\to\infty}\|f\|_{L^{p_k}} \le \|f\|_{L^\infty}$. Thus the limit is established. □

(c) Find $f \in \bigcap_{p\in\mathbb{N}} L^p(\Omega,\mu)$, where $\mu(\Omega) < +\infty$, with $f \notin L^\infty(\Omega,\mu)$, i.e., show that the result from part (a) does not hold true without the assumption $\sup\limits_{p\in\mathbb{N}}\|f\|_{L^p} < +\infty$.

Solution: For $f(x) = -\log(x)$, we clearly have $f \in L^p((0,1),\mathcal{L}^1)$ but $f \notin L^\infty$. □

Exercise 12.6.

Let $(x_{n,m})_{(n,m)\in\mathbb{N}^2} \subset [0,+\infty]$ be a sequence parametrized by $\mathbb{N}^2$. Show that

$$\sum_{(n,m)\in\mathbb{N}^2} x_{n,m} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{n,m} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x_{n,m}.$$

Remark. Given a sequence $(x_\alpha)_{\alpha\in A} \subset [0,+\infty]$ parametrized by an arbitrary set A, we define

$$\sum_{\alpha\in A} x_\alpha := \sup_{F\subset A \text{ finite}} \sum_{\alpha\in F} x_\alpha.$$

Solution: We show that $\sum_{(n,m)\in\mathbb{N}^2} x_{n,m} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{n,m}$, then the other equality follows analogously. Let $F \subset \mathbb{N}^2$ be any finite set, then there exists $N \in \mathbb{N}$ such that $F \subset \{0,1,\ldots,N\} \times \{0,1,\ldots,N\}$. Hence we get that

$$\sum_{(n,m)\in F} x_{n,m} \le \sum_{n=0}^{N}\sum_{m=0}^{N} x_{n,m} \le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{n,m}.$$

Taking the supremum over all $F \subset \mathbb{N}^2$, we thus get that $\sum_{(n,m)\in\mathbb{N}^2} x_{n,m} \le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{n,m}$. Let us now prove the reversed inequality. It is sufficient to show that $\sum_{n=0}^{N}\sum_{m=0}^{\infty} x_{n,m} \le \sum_{(n,m)\in\mathbb{N}^2} x_{n,m}$ for all $N \in \mathbb{N}$. Note that

$$\sum_{n=0}^{N}\sum_{m=0}^{\infty} x_{n,m} = \lim_{M\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{M} x_{n,m} = \lim_{M\to\infty}\sum_{(n,m)\in\{0,\ldots,N\}\times\{0,\ldots,M\}} x_{n,m} \le \sum_{(n,m)\in\mathbb{N}^2} x_{n,m},$$

which concludes the proof. □