Exercise 13.1.
Let $f \in L^p(\mathbb{R}, \lambda)$, where λ is the Lebesgue measure. By means of Fubini's Theorem, show that the following equality holds:

$$\int_{\mathbb{R}} |f(x)|^p dx = p \int_0^\infty y^{p-1} \lambda(\{x \in \mathbb{R} : |f(x)| \geq y\}) dy.$$

Hint: $|f(x)|^p = \int_0^{|f(x)|} p y^{p-1} dy$.

Remark. Compare with Exercise 10.4. In that case there was an underlying Fubini-type argument in the proof. This time we can use Fubini's Theorem and get a straightforward proof.

Solution: It is easy to see that $|f(x)|^p = \int_0^{|f(x)|} p y^{p-1} dy$. Therefore, using Fubini's Theorem in the second line (to change the order of integration), we get

$$\begin{aligned}
\int_{\mathbb{R}} |f(x)|^p dx &= \int_{\mathbb{R}} \left(\int_0^{|f(x)|} p y^{p-1} dy \right) dx = p \int_{\mathbb{R}} \left(\int_{\mathbb{R}} y^{p-1} \chi_{[0, |f(x)|]}(y) dy \right) dx \\
&= p \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_{\{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq |f(x)|\}}(x, y)\, dx \right) y^{p-1} dy \\
&= p \int_{\mathbb{R}} \lambda(\{x \in \mathbb{R} : |f(x)| \geq y\}) \chi_{[0,+\infty)}(y) y^{p-1} dy \\
&= p \int_0^\infty y^{p-1} \lambda(\{x \in \mathbb{R} : |f(x)| \geq y\}) dy.
\end{aligned}$$

□

Exercise 13.2.
Define the function $f : [0, 1]^2 \to \mathbb{R}$ as

$$f(x, y) := \begin{cases} y^{-2} & \text{if } 0 < x < y < 1, \\ -x^{-2} & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Is this function summable with respect to the Lebesgue measure?

Solution: We want to prove that f is not summable. Suppose it were summable. Then, we could change the order of integration thanks to Fubini's Theorem. However, this leads to a contradiction since

$$\int_0^1 \int_0^1 f(x, y) dx dy = \int_0^1 \left(\int_0^y \frac{1}{y^2} dx - \int_y^1 \frac{1}{x^2} dx \right) dy = 1$$

and

$$\int_0^1 \int_0^1 f(x, y) dy dx = \int_0^1 \left(\int_0^x -\frac{1}{x^2} dy + \int_x^1 \frac{1}{y^2} dy \right) dx = -1 \, .$$

□

Exercise 13.3.
Let $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^n)$ and, for all $h \in \mathbb{R}^n$, consider the function $\tau_h \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\tau_h(x) = x + h$. Show that

$$\|f \circ \tau_h - f\|_{L^p} \to 0 \qquad \text{as } h \to 0.$$

Hint: use the density of continuous and compactly supported functions in L^p (Theorem 3.7.15 in the Lecture Notes).

Solution: Fix $\varepsilon > 0$, then by Theorem 3.7.15 there exists $g \in C_c^0(\mathbb{R}^n)$ such that $\|f - g\|_{L^p} < \varepsilon/3$. Define the compact set $K = \{x \in \mathbb{R}^n \mid d(x, \operatorname{supp}(g)) \leq 1\}$, then for $|h| \leq 1$ we have

$$\|g \circ \tau_h - g\|_{L^p}^p = \int_K |g(x+h) - g(x)|^p dx \leq \mathcal{L}^n(K) \sup_{|x-y| \leq h} |g(x) - g(y)|.$$

Therefore, using that g is uniformly continuous, there exists $r > 0$ such that $\|g \circ \tau_h - g\|_{L^p} < \varepsilon/3$ for all $|h| \leq r$. Hence, for all $|h| \leq r$, we have

$$\|f \circ \tau_h - f\|_{L^p} \leq \|f \circ \tau_h - g \circ \tau_h\|_{L^p} + \|g \circ \tau_h - g\|_{L^p} + \|g - f\|_{L^p} < \varepsilon,$$

which proves what we wanted by arbitrariness of ε. □

Exercise 13.4.
We say that a family $(\varphi_\varepsilon)_{\varepsilon > 0}$ of functions in $L^1(\mathbb{R}^n)$ is an *approximate identity* if:

1. $\varphi_\varepsilon \geq 0$ and $\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = 1$ for all $\varepsilon > 0$;
2. for all $\delta > 0$ we have that $\int_{\{|x| \geq \delta\}} \varphi_\varepsilon(x) dx \to 0$ as $\varepsilon \to 0$.

(a) Given $\varphi \in L^1(\mathbb{R}^n)$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi(x) dx = 1$, define $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\varepsilon^{-1} x)$ for all $\varepsilon > 0$. Show that $(\varphi_\varepsilon)_{\varepsilon > 0}$ is an approximate identity.

Solution: Obviously we have that $\varphi_\varepsilon \geq 0$. Moreover

$$\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = \int_{\mathbb{R}^n} \varphi(\varepsilon^{-1}(x)) \varepsilon^{-n} dx = \int_{\mathbb{R}^n} \varphi(y) dy = 1,$$

where we made the change of variable $y = \varepsilon^{-1} x$ and we used the fact that $\mathcal{L}^n(\varepsilon^{-1} A) = \varepsilon^{-n} \mathcal{L}^n(A)$ for all $\mathcal{L}^n$-measurable sets A. Fix now $\delta > 0$, using the same change of variable we get

$$\int_{\{|x| \geq \delta\}} \varphi_\varepsilon(x) dx = \int_{\{|x| \geq \delta\}} \varphi(\varepsilon^{-1}(x)) \varepsilon^{-n} dx = \int_{\{|y| \geq \varepsilon^{-1}\delta\}} \varphi(y) dy,$$

which converges to 0 by the Dominated Convergence Theorem, since the functions $\varphi \chi_{\{|y| \geq \varepsilon^{-1}\delta\}}$ converge pointwise to zero almost everywhere and are dominated by the $\mathcal{L}^n$-summable function φ. □

Let $(\varphi_\varepsilon)_{\varepsilon > 0} \subset L^1(\mathbb{R}^n)$ be an approximate identity. Show that the following statements hold.

(b) If $f \in L^\infty(\mathbb{R}^n)$ is continuous at $x_0 \in \mathbb{R}^n$, then $f*\varphi_\varepsilon$ is continuous and $(f*\varphi_\varepsilon)(x_0) \to f(x_0)$ as $\varepsilon \to 0^+$.

Solution: Let us first prove that $f * \varphi_\varepsilon$ is continuous. Note that, for all $h \in \mathbb{R}^n$, we have

$$(f * \varphi_\varepsilon)(x+h) = \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x+h-y)dy = \int_{\mathbb{R}^n} f(y)(\varphi_\varepsilon \circ \tau_h)(x-y)dy = (f * (\varphi_\varepsilon \circ \tau_h))(x).$$

Hence, using Corollary 4.4.6 (ii) to the functions $f \in L^\infty(\mathbb{R}^n)$ and $\varphi_\varepsilon \circ \tau_h - \varphi_\varepsilon \in L^1(\mathbb{R}^n)$, we get

$$|(f * \varphi_\varepsilon)(x+h) - (f * \varphi_\varepsilon)(x)| = |(f * (\varphi_\varepsilon \circ \tau_h - \varphi_\varepsilon))(x)| \leq \|f\|_{L^\infty} \|\varphi_\varepsilon \circ \tau_h - \varphi_\varepsilon\|_{L^1},$$

which converges to 0 as $h \to 0$ thanks to Exercise 13.3. This proves that $f * \varphi_\varepsilon$ is continuous.

Given $\delta > 0$, by continuity of f at x_0, there exists $r > 0$ such that $|f(x_0 - y) - f(x_0)| < \delta$ for all $|y| < r$. Hence, using that $\int_{\mathbb{R}^n} \varphi_\varepsilon = 1$, we get

$$\begin{aligned}
|(f * \varphi_\varepsilon)(x_0) - f(x_0)| &\leq \int_{\mathbb{R}^n} |f(x_0 - y) - f(x_0)|\varphi_\varepsilon(y)dy \\
&= \int_{\{|y|<r\}} |f(x_0 - y) - f(x_0)|\varphi_\varepsilon(y)dy + \int_{\{|y|\geq r\}} |f(x_0 - y) - f(x_0)|\varphi_\varepsilon(y)dy \\
&\leq \delta + 2\|f\|_{L^\infty} \int_{\{|y|\geq r\}} \varphi_\varepsilon(y)dy,
\end{aligned}$$

which converges to δ as $\varepsilon \to 0$ be definition of approximate identity. This concludes the proof by arbitrariness of δ. □

(c) If $f \in L^\infty(\mathbb{R}^n)$ is uniformly continuous, then $f * \varphi_\varepsilon \xrightarrow{L^\infty} f$ as $\varepsilon \to 0^+$.

Solution: The solution works the same as the one of part (b) using that, given $\delta > 0$, there exists $r > 0$ such that $|f(x - y) - f(x)| < \delta$ for all $|y| < r$, where r does not depend on x. □

(d) If $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^n)$, then $f * \varphi_\varepsilon \xrightarrow{L^p} f$ as $\varepsilon \to 0^+$.

Hint: use Hölder's inequality and keep in mind Exercise 13.3 and part (b).

Solution: First note that, by Corollary 4.4.6 (ii), $f * \varphi_\varepsilon \in L^p(\mathbb{R}^n)$. Now, using that $\int_{\mathbb{R}^n} \varphi_\varepsilon = 1$ and Hölder inequality, we get

$$\begin{aligned}
|(f * \varphi_\varepsilon)(x) - f(x)|^p &\leq \left| \int_{\mathbb{R}^n} (f(x-y) - f(x))\varphi_\varepsilon(y)dy \right|^p \\
&= \left| \int_{\mathbb{R}^n} (f(x-y) - f(x))\varphi_\varepsilon(y)^{1/p}\varphi_\varepsilon(y)^{1/p'}dy \right|^p \\
&\leq \left(\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \varphi_\varepsilon(y)dy \right) \left(\int_{\mathbb{R}^n} \varphi_\varepsilon(y)dy \right)^{p/p'} \\
&= \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \varphi_\varepsilon(y)dy.
\end{aligned}$$

Then we integrate over $\mathbb{R}^n$ and use Tonelli's theorem to get

$$\begin{aligned}\int_{\mathbb{R}^n} |(f*\varphi_\varepsilon)(x) - f(x)|^p dx &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)-f(x)|^p \varphi_\varepsilon(y) dy dx \\ &= \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\left(\int_{\mathbb{R}^n} |f(x-y)-f(x)|^p dx\right) dy = \int_{\mathbb{R}^n} \varphi_\varepsilon(y) \|f\circ\tau_{-y} - f\|_{L^p}^p dy.\end{aligned}$$

Now denote by $g\colon \mathbb{R}^n \to [0,+\infty)$ the function $g(y) = \|f\circ\tau_{-y} - f\|_{L^p}^p$. Observe that, by Exercise 13.3, the function g is continuous. Moreover $g(y) \le 2^p \|f\|_{L^p}^p$, hence $g \in L^\infty(\mathbb{R}^n)$. Therefore we can use part (b) to obtain that $(g*\varphi_\varepsilon)(0) \to g(0) = 0$ as $\varepsilon \to 0$. However note that this concludes the proof since $\int_{\mathbb{R}^n} \varphi_\varepsilon(y)\|f\circ\tau_{-y} - f\|_{L^p}^p dy = (g*\varphi_\varepsilon)(0)$. □

Exercise 13.5.
Compute the following limits:

(a)

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx}{(1+x)^n}\, dx.$$

Solution: It is clear that the constant function 1, which is summable on $[0,1]$, dominates the sequence. Moreover, for all $x>0$ the integrand tends to 0 as $n\to\infty$. Therefore, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx}{(1+x)^n}\, dx = \int_0^1 \lim_{n\to\infty} \frac{1+nx}{(1+x)^n}\, dx = \int_0^1 0\, dx = 0.$$ □

(b)

$$\lim_{n\to\infty} \int_0^1 \frac{x\log x}{1+n^2x^2}\, dx.$$

Solution: The integrand is clearly bounded above by the function $x|\log x|$, which is bounded on $(0,1)$ and therefore summable. Moreover, the sequence of integrands tends to 0 away from $x=0$. Therefore, as above, the limit of the integrals is 0.

Exercise 13.6.
Let $I = [0,1]$ and consider the function

$$f : I^3 \to [0,\infty], \quad f(x,y,z) := \begin{cases} \frac{1}{\sqrt{|y-z|}}, & \text{if } y \ne z, \\ \infty, & \text{if } y = z. \end{cases}$$

Show that $f \in L^1(I^3, \mathcal{L}^3)$.

Solution: Note that $f \geq 0$ and that f is continuous outside the closed set $\{y = z\}$. This shows that f is Lebesgue-measurable. We first apply Fubini's theorem twice:

$$\int_{I^3} f(x,y,z) d\mathcal{L}^3(x,y,z) = \int_I \left(\int_{I^2} f(x,y,z) d\mathcal{L}^2(y,z) \right) d\mathcal{L}^1(x)$$
$$= \int_I \left(\int_I \left(\int_I f(x,y,z) d\mathcal{L}^1(y) \right) d\mathcal{L}^1(z) \right) d\mathcal{L}^1(x).$$

Now we compute the inner integral for x, z fixed:

$$\begin{aligned}\int_I f(x,y,z)\, d\mathcal{L}^1(y) &= \int_{I\setminus\{z\}} \frac{1}{\sqrt{|y-z|}}\, d\mathcal{L}^1(y) \\ &= \int_0^z \frac{1}{\sqrt{z-y}}\, d\mathcal{L}^1(y) + \int_z^1 \frac{1}{\sqrt{y-z}}\, d\mathcal{L}^1(y) \\ &= \left[-2\sqrt{z-y}\right]_{y=0}^{y=z} + \left[2\sqrt{y-z}\right]_{y=z}^{y=1} \\ &= 2\sqrt{z} + 2\sqrt{1-z}.\end{aligned}$$

Therefore for each $x \in I$ we have

$$\int_{I^2} f(x,y,z)\, d\mathcal{L}^2(y,z) = \int_I 2\sqrt{z} + 2\sqrt{1-z}\, d\mathcal{L}^1(z) = \frac{8}{3},$$

and finally we get

$$\int_{I^3} |f(x,y,z)|\, d\mathcal{L}^3(x,y,z) = \int_{I^3} f(x,y,z)\, d\mathcal{L}^3(x,y,z) = \int_I \frac{8}{3}\, d\mathcal{L}^1(x) = \frac{8}{3} < \infty,$$

which shows that $f \in L^1(I^3, \mathcal{L}^3)$.

Exercise 13.7.

Find a sequence of Lebesgue-measurable functions $f_n : [0,1] \to \mathbb{R}$ such that $\{f_n(x)\}_{n\in\mathbb{N}}$ is unbounded for any $x \in [0,1]$ but $f_n \to 0$ in measure.

Solution: For $n \in \mathbb{Z}^+$ and $k \in \{1, \ldots, n\}$, let $g_n^k(x) = n\chi_{\left[\frac{k-1}{n}, \frac{k}{n}\right]}(x)$ and look at the sequence $g_1^1, g_2^1, g_2^2, g_3^1, g_3^2, g_3^3, \ldots$. It is clear that

$$\mathcal{L}^1(\{x \in [0,1] \mid |g_n^k(x)| > \varepsilon\}) = \mathcal{L}^1\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \frac{1}{n} \xrightarrow{n\to\infty} 0.$$

for any $\varepsilon > 0$, which shows that the sequence converges to the function 0 in measure. On the other hand, given $x \in [0,1]$, for each $n \in \mathbb{Z}^+$ we can choose $k \in \{1, \ldots, n\}$ such that $nx \in [k-1, k]$, which means that $g_n^k(x) = n$. This implies that the sequence $\{g_n^k(x)\}_{n,k}$ is unbounded.