Complex Analysis

(Funktionentheorie)

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Organization:

(1) Home page

metaphor.ethz.ch/x/2022/hs/401-2303-00L/

(2) Forum

forum.math.ethz.ch/c/autumn-2022/complex-analysis-d-math-d-phys/131

(3) Book

E. Stein & R. Shakarchi, "Complex analysis" (Princeton, 2003)

(4) Exercises / bonus system

→ see the home page / forum

Chapter I

Introduction

1 – What is complex analysis?

At first, it seems to be just about taking Analysis I and replacing $\mathbb{R}$ by $\mathbb{C}$ in the definitions... But then remarkable things happen!

For instance, here are a few statements where complex numbers are nowhere to be seen:

(1) The integral

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^n}$$

is equal to $\dfrac{\pi}{2^{2n-2}} \dfrac{(2n-2)!}{((n-1)!)^2}$.

(2) The Taylor expansion around 0 (resp. 1) of $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\begin{cases} f(0) = 1 \\ f(x) = \dfrac{x}{e^x - 1}, & x \neq 0 \end{cases}$$

has radius of convergence 2π (resp. $\sqrt{1+4\pi^2}$)

(3) The number $\pi(x)$ of prime numbers $p \leq x$ satisfies $\pi(x) \underset{x \to \infty}{\sim} \frac{x}{\log x}$.

(i.e. $\lim_{x \to +\infty} \pi(x) / {}^{x}/_{\log x} = 1$; the "Prime Number Theorem")

In <u>all these cases</u> (and many more), it turns out that the simplest (fastest) way to prove these facts is by means of <u>complex-valued functions of a complex variable</u>. Moreover (and this is even more important) the methods and ideas that are involved are extremely <u>robust</u>: it is not just <u>one</u> problem of the kinds above that we can solve, but <u>many</u> of them.

In this course, we will see how to do all this (but maybe not a full proof of the third example, by lack of time...)

2 - A few amazing facts

The main definition of complex analysis is that of a holomorphic function f, say defined on all of $\mathbb{C}$: this is a function $f: \mathbb{C} \longrightarrow \mathbb{C}$ which is "differentiable in the complex sense": for each $z_0 \in \mathbb{C}$, the limit

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists (in $\mathbb{C}$). Now it turns out that this implies the following properties:

(i) f is infinitely differentiable (so f', f'', ..., $f^{(n)}$, ..., exist and are all holomorphic).

(ii) In fact, the Taylor expansion of f

around any $z_0 \in \mathbb{C}$ converges everywhere to f (holomorphic functions are analytic).

(iii) Moreover, for any $z_0 \in \mathbb{C}$ and any radius $r > 0$, the value of f at z_0 is given by

$$f(z_0) = \frac{1}{2i\pi} \int_{\circlearrowleft} \frac{f(z)}{z - z_0} \, dz$$

("Cauchy Integral Formula")

(iv) And if g is another holomorphic function which coincides with f on an arbitrarily small disc (or even just on a convergent sequence (z_n)) then we have $f = g$ <u>everywhere</u> (Analytic Continuation Principle).

(v) And if f happens to be bounded, then f is constant (Liouville's Theorem); if f is bounded by a polynomial, it <u>is</u> a polynomial.

(vi) And the maximum of $|f(z)|$ for z in a closed disc is always attained on the boundary (Maximum Modulus Principle).

... and many more!

Example. One of many proofs of the "fundamental theorem of algebra" using complex analysis: let $f \in \mathbb{C}[X]$ be a non-zero polynomial; if f has no zeros in $\mathbb{C}$ then (we will see) $1/f$ is holomorphic on $\mathbb{C}$; but $1/f$ is also bounded (easy to see), so by Liouville's Theorem, $1/f$ is constant, and so f has degree 0.

3 _ Reminders: complex numbers

Before we begin, some reminders of notation and terminology.

<u>Complex numbers</u> $z \in \mathbb{C}$ are usually written $z = a + ib$, and unless specified otherwise, this <u>implicitly means</u> that $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$. We also have

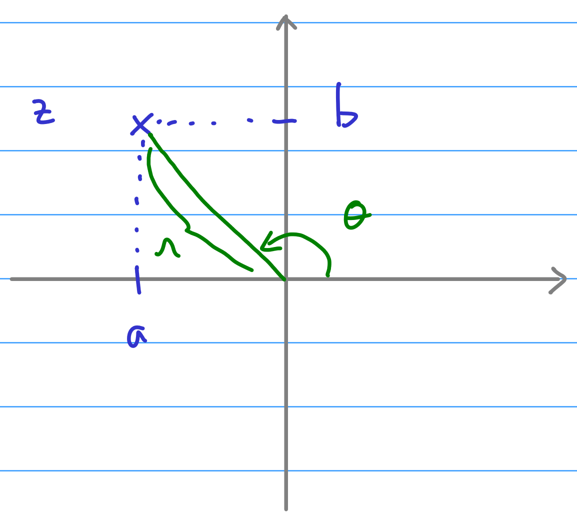
$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$$

and especially $z \in \mathbb{R}$ (i.e. $b = 0$) $\iff z = \bar{z}$

$(z \in i\mathbb{R} \iff z = -\bar{z})$

The polar-coordinate representation is



$$z = r(\cos\theta + i\sin\theta)$$

$$= r e^{i\theta}$$

with $r \geqslant 0$, $\theta \in \mathbb{R}$.

It is unique if $z \neq 0$

and $\theta \in [0, 2\pi[$ (or $\theta \in]-\pi, \pi]$, ...)

To go back and forth, one has the formulas:

$$\begin{cases} r = |z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2} \\ \tan(\theta) = \dfrac{b}{a} = \dfrac{1}{i}\,\dfrac{z-\bar{z}}{z+\bar{z}}, \quad \text{if } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{cases}$$

and

$$\begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases}$$

The complex exponential $\exp : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto e^z$

is defined by the power series

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}.$$

[exp is holomorphic]

It satisfies $e^{z+w} = e^z e^w$ for all z, w in $\mathbb{C}$,

and $\left(e^z = 1 \iff \exists\, k \in \mathbb{Z},\ z = 2ik\pi\right)$.

4. Reminders: topology

Because we look at functions defined on $\mathbb{C}$ (or subsets), we must handle rather more compli-cated issues than in Analysis I when it comes to understanding the "good" sets on which to define functions.

First, we will denote

$$D_r(z) \quad \text{or} \quad D(z;r)$$

the open disc of radius r centered at z:

$$D_r(z) = \{ w \in \mathbb{C} \mid |w-z| < r \}$$

and

$$\overline{D}_r(z) \quad \text{or} \quad \overline{D}(z;r)$$

the closed disc:

$$\overline{D}_r(z) = \{ w \in \mathbb{C} \mid |w-z| \leq r \}$$

The circle $\overline{D}_r(z) \underbrace{-}_{\text{set difference}} D_r(z)$ is the "boundary" of $\overline{D}_r(z)$:

$$C_r(z) = \{ w \in \mathbb{C} \mid |w - z| = r \}.$$

Type of set	Example
Open set $U \subset \mathbb{C}$: $\forall z \in U, \exists r > 0, D_r(z) \subset U$	$\emptyset$, $\mathbb{C}$, $D_r(z)$ $\mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$
Closed set $G \subset \mathbb{C}$: $\mathbb{C} - G = \{ z \in \mathbb{C} \mid z \notin G \}$ is open $\Updownarrow$ $\forall (z_n)$, $z_n \in G$, if $z_n \to z$ then $z \in G$	$\emptyset$, $\mathbb{C}$, $\overline{D}_r(z)$, $C_r(z)$, $\mathbb{R}$, $[0,1] \times [0,1]$ $\{ z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0 \}$
Compact set $K \subset \mathbb{C}$: K is compact and bounded $(\exists M, \forall z \in K, \lvert z \rvert \leq M)$ $\Updownarrow$ $\forall (z_n)$, $z_n \in K$, there is a convergent subsequence (z_{n_k})	$\emptyset$, $\overline{D}_r(z)$, $C_r(z)$ $[0,1] \times [0,1]$ $\{z_n\} \cup \{z_\infty\}$ if $z_n \longrightarrow z_\infty$
Connected set $A \subset \mathbb{C}$: if $U_1, U_2 \subset \mathbb{C}$ are open, $A \subset U_1 \cup U_2$, $U_1 \cap U_2 \cap A = \emptyset$ then $U_1 \cap A = A$ or $U_2 \cap A = A$ $\iff$ any $f : A \longrightarrow \{0, 1\}$ continuous is <u>constant</u>	$\emptyset$, $\mathbb{C}$, $D_r(z)$ $\overline{D}_r(z)$, $C_r(z)$, $\mathbb{R}$ but <u>not</u> $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R} \cup D_{1/2}(i)$

5 _ Reminders : limits

We recall how to compute limits in $\mathbb{C}$.

(1) A sequence (z_n) with $z_n = a_n + i b_n$ converges to $z = a + ib$ if any of the following equivalent conditions is true :

(i) $a_n \longrightarrow a$ and $b_n \longrightarrow b$ in $\mathbb{R}$

(ii) $|z_n - z| \longrightarrow 0$ in $\mathbb{R}$

(iii) $\forall \varepsilon > 0, \exists N, \forall m, n \geq N, |z_m - z_n| < \varepsilon$

(Cauchy Criterion)

(2) Let $A \subset \mathbb{C}$ be any subset and $f : A \longrightarrow \mathbb{C}$ any function. For $z_0 \in A$ and $w_0 \in \mathbb{C}$, we have

$$\lim_{\substack{z \to z_0 \\ z \in A}} f(z) = w_0$$

if any of the following holds:

(i) if (z_n) is a sequence in A with $z_n \longrightarrow z_0$, then $f(z_n) \longrightarrow w_0$.

(ii) $\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall z \in A,$

$$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon .$$

(3) Finally, for any subset $A \subset \mathbb{C}$ and any $f : A \longrightarrow \mathbb{C}$, the function f is <u>continuous on A</u> if and only if

$$\forall z_0 \in A, \quad \lim_{\substack{z \to z_0 \\ z \in A}} f(z) = f(z_0).$$