Chapter II

Holomorphic functions

1 - Definition and first examples

Definition - Let $U \subset \mathbb{C}$ be an open set. A function $f : U \longrightarrow \mathbb{C}$ is holomorphic on U if, for all $z_0 \in U$, the limit

$$\lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(z_0+h) - f(z_0)}{h}$$

put $h = z - z_0$

exists (in $\mathbb{C}$).

This limit is called the (complex) derivative of f at z_0, and is denoted $f'(z_0)$.

Remark - One can also speak of f holomorphic at z_0 only, with the obvious definition.

Proposition - Let $U \subset \mathbb{C}$ be open.

(1) The set $\mathcal{H}(U)$ of holomorphic functions on U is a vector space (subspace of the

space of all $f : U \longrightarrow \mathbb{C}$), with

$$(\lambda f + \mu g)' = \lambda f' + \mu g'$$

(λ, μ in $\mathbb{C}$, f, g in $\mathcal{H}(U)$). The elements of $\mathcal{H}(U)$ are continuous on U.

(2) If f, g are in $\mathcal{H}(U)$, then fg also and $(fg)' = f'g + fg'$; if $g(z) \neq 0$ for all $z \in U$, then $f/g \in \mathcal{H}(U)$ with

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

(3) Polynomial functions on U are in $\mathcal{H}(U)$, with

$$\left(\sum_{i=0}^{d} a_i z^i\right)' = \sum_{i=1}^{d} i a_i z^{i-1}.$$

meaning the function $z \longmapsto \sum_{i=0}^{d} a_i z^i$

Proof. All these statements are proved exactly in the same way as for functions defined on intervals in $\mathbb{R}$. Just to illustrate this, here is the proof of (2).

Consider $z_0 \in U$. For $z \in U$, write

$$f(z)g(z) - f(z_0)g(z_0) = f(z)\big(g(z) - g(z_0)\big) + g(z_0)\big(f(z) - f(z_0)\big)$$

then divide by $z - z_0$ for $z \neq z_0$:

$$\frac{fg(z) - fg(z_0)}{z - z_0} = f(z)\,\frac{g(z) - g(z_0)}{z - z_0} + g(z_0)\,\frac{f(z) - f(z_0)}{z - z_0}$$

and let $z \to z_0$: since f is continuous (by (1)) we have $f(z) \to f(z_0)$, so by definition

$$\frac{fg(z) - fg(z_0)}{z - z_0} \xrightarrow[z \neq z_0]{z \to z_0} f(z_0)\,g'(z_0) + g(z_0)\,f'(z_0)$$

as claimed.

$\square$

This leads us naturally to the most important examples:

[Th. 2.6 in Stein-Shakarchi]

Proposition. Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers such that the power series

$$\sum_{n \geq 0} a_n z^n$$

has radius of convergence $r > 0$. Let $w_0 \in \mathbb{C}$

and let $f : D_r(w_0) \longrightarrow \mathbb{C}$ be defined by

$$f(z) = \sum_{n \geq 0} a_n (z - w_0)^n.$$

Then $f \in \mathcal{H}(D_r(w_0))$ and

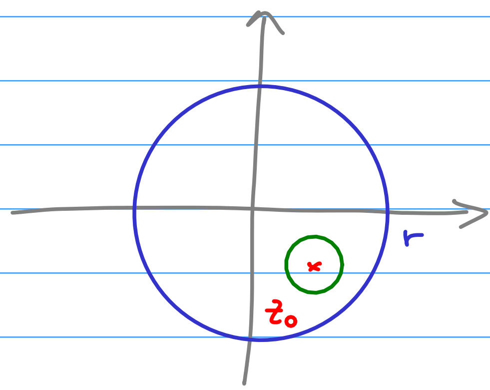
$$f'(z) = \sum_{n \geq 1} n a_n (z - w_0)^{n-1}$$

for $z \in D_r(w_0)$. In particular, $f' \in \mathcal{H}(D_r(w_0))$.

<u>Proof</u> - This is also "the same" as Analysis I, but because it is so important, we explain the argument. We assume that $w_0 = 0$.

For $N \geq 1$, we write $f = f_N + g_N$ where $f_N(z) = \sum_{n=0}^{N} a_n z^n$. The idea now is

(i) f_N is a polynomial, so holomorphic



(ii) g_N is <u>uniformly</u> small when N is large.

More precisely, for $z_0 \in D_r(0)$, pick $s < r$ with $|z_0| < s$ and for $z \in D_s(0)$, write

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{f_N(z)-f_N(z_0)}{z-z_0} + \frac{g_N(z)-g_N(z_0)}{z-z_0}$$

Now fix $\varepsilon > 0$. We show:

(a) For any N large enough, $\left|\boxed{}\right| \leq \varepsilon$ <u>for all $z \in D_s(0)$</u>.

(b) For any fixed N, for all z close enough to z_0

$$\left|\boxed{} - f_N'(z_0)\right| \leq \varepsilon$$

(c) For any N large enough, and any $z \in D_s(0)$,

$$\left| f_N'(z_0) - \sum_{n=1}^{+\infty} n a_n z_0^{n-1} \right| \leq \varepsilon .$$

Using (a), (c), we fix a <u>single</u> N (depending on ε) such that both properties are true. Then (b), for <u>this</u> N, implies that there exists $\delta > 0$ such that the condition $|z - z_0| < \delta$ ("z close enough to z_0") implies

$$\left| \frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=1}^{+\infty} n a_n z_0^{n-1} \right| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

(using the triangle inequality).

Now for checking (a), (b), (c):

(a) and (c) are similar; let's do (a):

$$\frac{g_N(z) - g_N(z_0)}{z - z_0} = \sum_{n=N+1}^{\infty} a_n \frac{z^n - z_0^n}{z - z_0}$$

but

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \dots + z z_0^{n-2} + z_0^{n-1}$$

so $\left| \frac{z^n - z_0^n}{z - z_0} \right| \leq n s^{n-1}$ (since $|z_0| < s$, $z \in D_s(0)$)

and therefore

$$\left| \frac{g_N(z) - g_N(z_0)}{z - z_0} \right| \leq \sum_{n=N+1}^{+\infty} n |a_n| s^{n-1}.$$

By assumption on the radius of convergence of f, this tends to 0 as $N \to +\infty$, which gives (a) since z does not appear on RHS.

Finally, (b) is just the definition of $f'_N(z_0)$.

$\square$

Examples. Now we have many examples!

(1) $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic on $\mathbb{C}$.

(2) The Bessel function

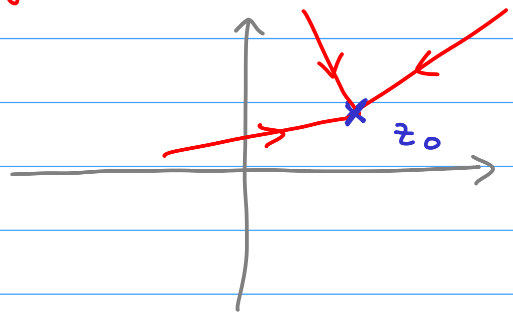
$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,(n+r)!} \left(\frac{z}{2}\right)^{2r}$$

is holomorphic on $\mathbb{C}$ (for all fixed integers n).

(3) But here is an important counterexample: let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be $f(z) = \bar{z}$. Then f is <u>not</u> holomorphic; in fact, its derivative never exists: let $z_0 \in \mathbb{C}$, and $h \in \mathbb{C}^\times$; then

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\bar{h}}{h},$$

which does <u>not</u> converge as $h \longrightarrow 0$ (because taking $h \longrightarrow 0$ in "direction θ", $h = |h| e^{i\theta}$, gives a limit $e^{-2i\theta}$ which <u>depends on θ</u>).



2 – <u>Holomorphicity and differentiability</u>

Let $U \subset \mathbb{C}$ be an open set. A function

$$f : U \longrightarrow \mathbb{C}$$

can also be interpreted as a function of two real variables x, y, namely

$$\tilde{f} : (x, y) \longmapsto f(x+iy)$$

and we can further write

$$\tilde{f}(x,y) = u(x,y) + i\,v(x,y)$$

where $u, v : U \longrightarrow \mathbb{R}$ are the real and imaginary parts.

Recall now that for any $F : U \longrightarrow \mathbb{C}$, we can ask whether it is <u>differentiable</u> on U or not (or differentiable at a point). Recall further that this means that F can be approximated close de $(x_0, y_0) \in U$ by a <u>linear map</u>, or matrix

$$A_{(x_0,y_0)} = J_F(x_0, y_0).$$

Jacobi matrix

Writing $F = u + iv$ (i.e. $F(x,y) = (u(x,y), v(x,y))$) the matrix is

$$A_{(x_0,y_0)} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}.$$

It is now natural to ask what relations exist between this multivariable calculus version of

differentiability and holomorphy.

[cf. Stein & Shakarchi, Prop. 2.3, Th. 2.4]

Proposition. The following properties are equivalent:

(1) $f : U \longrightarrow \mathbb{C}$ is <u>holomorphic</u> on U

(2) $\tilde{f} = (u, v)$ is differentiable on U <u>and</u> for all $z_0 \in U$, the matrix A_{z_0} is of the form

$$A_{z_0} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \in M_2(\mathbb{R}).$$

(3) The functions $u, v : U \longrightarrow \mathbb{R}$ are differentiable and satisfy

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases}$$

("Cauchy–Riemann equations")

When this is the case, we have for all $z_0 \in U$ the relations

$$\begin{cases} f'(z_0) = \dfrac{\partial u}{\partial x}(z_0) + \dfrac{1}{i}\dfrac{\partial u}{\partial y}(z_0), \\ \det J(z_0) = |f'(z_0)|^2. \end{cases}$$

Proof. $(1) \Rightarrow (2)$: let $\begin{cases} z_0 = (x_0, y_0) \in U \\ h = (h_1, h_2) \in \mathbb{C} \end{cases}$; then

$$f(z_0 + h) = f(z_0) + f'(z_0) h + h\,\varepsilon(h)$$

where $\lim_{h \to 0} \varepsilon(h) = 0$ since f is holomorphic at z_0. By definition, this means that $\tilde{f}$ is differentiable at z_0 with differential A_{z_0} such that $A_{z_0} h = f'(z_0) h$. Let $f'(z_0) = \alpha + i\beta$.

Then $f'(z_0) h = (\alpha + i\beta)(h_1 + i h_2)$

$$= \alpha h_1 - \beta h_2 + i(\beta h_1 + \alpha h_2)$$

which means that $A_{z_0} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

(2) $\Rightarrow$ (3) : we know that if $\tilde{f}$ is differentiable, so are $u = \mathrm{Re}(\tilde{f})$ and $v = \mathrm{Im}(\tilde{f})$; the Jacobi matrix is $A_{z_0} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$,

so by comparing we get

$$\begin{cases} \alpha = \partial_x u = \partial_y v \\ \beta = \partial_x v = -\partial_y u \end{cases}$$

which are the Cauchy-Riemann equations.

(3) $\Rightarrow$ (1): let again $z_0 = x_0 + iy_0 \in U$, $h = (h_1, h_2) \in \mathbb{C}$.

The differentiability of u and v at z_0 mean that

$$u(z_0 + h) = u(z_0) + \partial_x u \, h_1 + \partial_y u \, h_2 + |h| \varepsilon_1(h)$$

at z_0

at z_0

$$v(z_0 + h) = v(z_0) + \partial_x v \, h_1 + \partial_y v \, h_2 + |h| \varepsilon_2(h)$$

where $\lim_{h \to 0} \varepsilon_1(h) = \lim_{h \to 0} \varepsilon_2(h) = 0$.

Multiply the second by i and add: we get

$$f(z_0 + h) = f(z_0) + (\partial_x u + i \partial_x v) h_1$$
$$+ (\partial_y u + i \partial_y v) h_2 + |h| \varepsilon(h)$$

with $\varepsilon(h) = \varepsilon_1(h) + i \varepsilon_2(h) \longrightarrow 0$ as $h \to 0$.

By the Cauchy-Riemann equations, the second/third terms become $(\partial_x u - i \partial_y u) h_1 + (\partial_y u + i \partial_x u) h_2$

$$= (\partial_x u - i \partial_y u)(h_1 + i h_2)$$

and hence $f'(z_0)$ exists and is equal to

$$(\partial_x u - i \partial_y u)(z_0).$$

This last formula is part of the final statement, and moreover from (2) we get

$$\det J_{z_0} = \alpha^2 + \beta^2$$
$$= |\alpha + i\beta|^2 = |f'(z_0)|^2.$$

$\square$

Notation - The following formal notation are extremely convenient and useful: whenever a function $f : U \longrightarrow \mathbb{C}$ has partial derivatives, we put

$$\begin{cases} \dfrac{\partial f}{\partial z} = \partial_z f = \dfrac{1}{2}\left(\partial_x f - i\,\partial_y f\right) \\ \dfrac{\partial f}{\partial \bar{z}} = \partial_{\bar{z}} f = \dfrac{1}{2}\left(\partial_x f + i\,\partial_y f\right). \end{cases}$$

$$\boxed{\begin{aligned} \partial_x f &= \partial_x u + i\partial_x v \\ \partial_y f &= \partial_y u + i\partial_y v \end{aligned}}$$

Then for f holomorphic, we have

$$\begin{aligned} \partial_{\bar{z}} f &= \frac{1}{2}\Big\{\left(\partial_x u + i\,\partial_x v\right) \\ &\qquad + i\left(\partial_y u + i\partial_y v\right)\Big\} \\ &= \frac{1}{2}\Big\{\left(\partial_x u - \partial_y v\right) + i\left(\partial_x v + \partial_y u\right)\Big\} \\ &= 0 \end{aligned}$$

by the Cauchy-Riemann equations, and

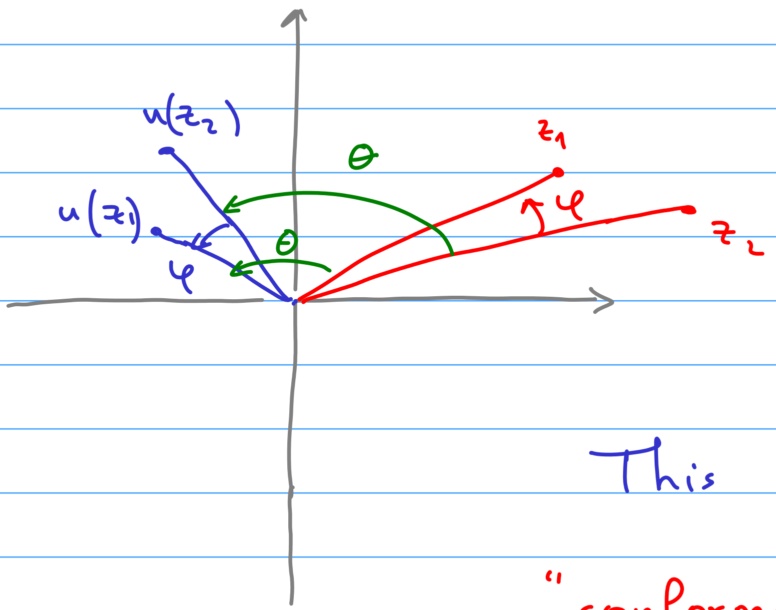
$$\partial_z f = f'$$

(similar computation).

Remark. What is special about a matrix of the form $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$? It is that if $\alpha + i\beta \neq 0$, then it represents a linear map

$$u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

which preserves angles and orientation (i.e. a combination of a rotation, by the angle θ such that $\alpha + i\beta = |\alpha + i\beta| e^{i\theta}$, and a dilation, by $|\alpha + i\beta|$.



$$|\alpha + i\beta| = \frac{|u(z_1)|}{|z_1|} = \frac{|u(z_2)|}{|z_2|}$$

This property (called "conformality") is the key to many geometric applications/properties of holomorphic maps.

3 _ Line integrals

One of the most important tool in the study of holomorphic functions is that of line integrals, i.e integrals of the form

$$\int_\gamma f(z)\,dz$$

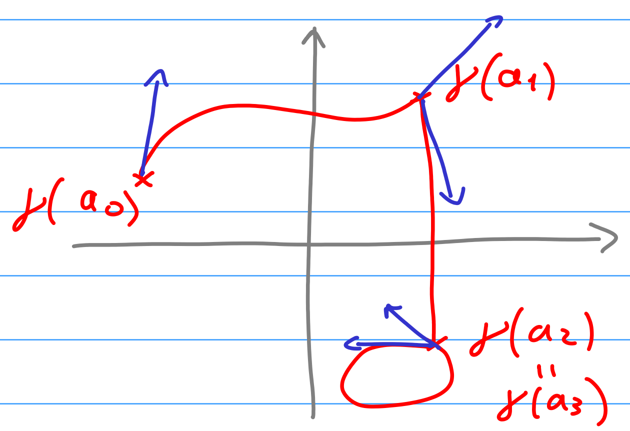
for some (oriented) curve $\gamma \subset \mathbb{C}$. These have been studied in Analysis I/II but we recall the main definitions and properties.

Definition.

(1) [Parameterized curve] A parameterized curve in $\mathbb{C}$ is a continuous map $\gamma : [a,b] \longrightarrow \mathbb{C}$ such that γ is piecewise C^1: there exists $n \geq 1$ and

$$a = a_0 < a_1 < \dots < a_{n-1} < a_n = b$$

such that γ restricted to $]a_i, a_{i+1}[$ is of class C^1, and has a right derivative at a_i and a left derivative at a_{i+1}, and moreover $\gamma'(t) \neq 0$ for all t (including left/right derivatives).



(2) [Closed curve] A γ as above is closed if $\gamma(a) = \gamma(b)$.

(3) [Reparameterization] Let γ be as above and let $\sigma : [c, d] \longrightarrow [a, b]$ be a C^1-function, bijective and with $\sigma'(t) > 0$ for all t. Then $\gamma \circ \sigma : [c, d] \longrightarrow \mathbb{C}$ is a <u>reparameterization</u> of γ. Intuitively, it represents the same geometric object (a curve in $\mathbb{C}$) as γ, but with a different parameterization.

When working with curves, we use often a specific parameterization, but the really significant notions / results are those which are <u>independent</u> of the choice of γ. The most important is the integral of a continuous function.

<u>Definition</u> [Line integral]

Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a curve. Let f be defined on a subset of $\mathbb{C}$ containing the image of γ, and assume f is continuous. The integral

of f along γ is
$$\int_\gamma f(z)\,dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\,\gamma'(t)\,dt.$$
[intuitively: $z=\gamma(t)$, $dz=\gamma'(t)dt$]

The crucial fact is:

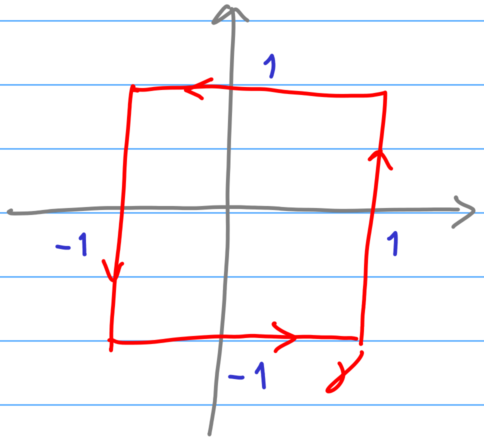
<u>Proposition</u>. The integral of f along γ is unchanged by a reparameterization.

<u>Proof</u>. Let $0 \leq i \leq n-1$ be given and c_i such that $\sigma(c_i) = a_i$. Then
$$\int_{c_i}^{c_{i+1}} f(\gamma(\sigma(s)))\,(\gamma\circ\sigma)'(s)\,ds = \int_{a_i}^{a_{i+1}} f(\gamma(t))\,\gamma'(t)\,dt$$
by the change of variable formula for integrals.

$\square$

<u>Example</u>. Because of this independance property, we often describe curves just by drawing them, when it is clear that a parameterization exists, with an indication of the orientation.

For instance, polygons are fine. For instance:



$$\gamma : [0, 4] \longrightarrow \mathbb{C}$$

and $\gamma(t) = 1 + i(2t - 1)$

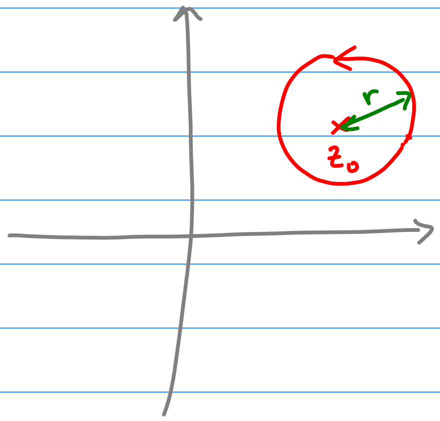
for $0 \leq t \leq 1$

$\gamma(t) = (1 - 2(t-1)) + i$

for $1 \leq t \leq 2$

etc...

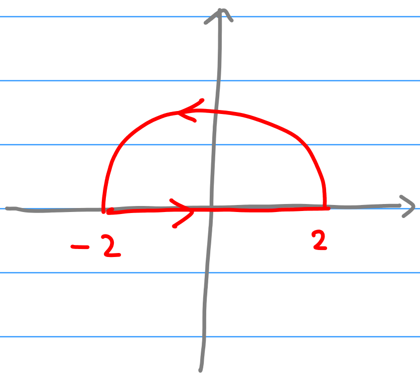
A circle can also be parameterized:



$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$$

$$t \longmapsto z_0 + r e^{it}$$

or a half-circle:



$$\gamma : [0, \pi + 1] \longrightarrow \mathbb{C}$$

$\gamma(t) = 2e^{it}, \quad 0 \leq t \leq \pi$

$\gamma(t) = -2 + 4(t - \pi), \quad \pi \leq t \leq \pi + 1.$

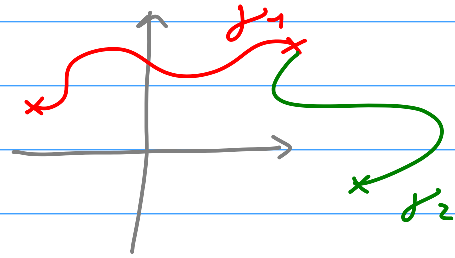
Proposition - [3.1 in Stein-Shakarchi] We have:

(1) $\displaystyle\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$

(2) $$\int_{\gamma^-} f = -\int_\gamma f$$ where γ^- is γ with reversed orientation

(3) $$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$ where

$\gamma_1 + \gamma_2$ is "first γ_1, then γ_2", when it makes sense



(4) We have

$$\left| \int_\gamma f(z)\,dz \right| \leq \sup_{z \in \gamma} |f(z)| \; \text{length}(\gamma)$$

where
$$\begin{cases} \sup\limits_{z\in\gamma} |f(z)| = \sup\limits_{t\in[0,b]} |f(\gamma(t))| \\ \text{length}(\gamma) = \sum\limits_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |\gamma'(t)|\,dt. \end{cases}$$

(which is independent of parametrization)

All these properties follow from those of the Riemann integral, for instance

$$\left| \int_\gamma f(z)\,dz \right| = \left| \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\,\gamma'(t)\,dt \right|$$
$$\leq \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq \sup_{t \in [a,b]} |f(\gamma(t))| \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |\gamma'(t)| dt.$$

4 - Primitives

Definition. (Primitive)

Let $U \subset \mathbb{C}$ be open, and $f : U \longrightarrow \mathbb{C}$ a function. We say that f admits a primitive F if $F \in \mathcal{H}(U)$ satisfies $F' = f$.

This is a natural definition. It leads to the following interesting fact:

Proposition. [Th. 3.2]

Let $U \subset \mathbb{C}$ be open, $f : U \longrightarrow \mathbb{C}$ continuous with a primitive $F \in \mathcal{H}(U)$. Let γ be any curve with image in U and joining $z_1 \in U$ to $z_2 \in U$. (i.e $\gamma : [a,b] \longrightarrow \mathbb{C}$ satisfies $\gamma(a) = z_1$, $\gamma(b) = z_2$)

Then

$$\int_\gamma f(z)\, dz = F(z_2) - F(z_1).$$

In particular, if γ is a closed curve, then

$$\int_\gamma f(z)\,dz = 0 .$$

Proof: Suppose first that $\gamma : [a,b] \longrightarrow \mathbb{C}$ is C^1 on $]a,b[$. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt .$$

This immediately suggest to say that this is

$$\int_a^b (F\circ\gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a))$$

and to apply the fundamental theorem of calculus. Let's check that this is justified (this is a basic compatibility between the real derivative of $F\circ\gamma$ and the complex derivative of F): according to Section 2, the function

$$F\circ\gamma : [a,b] \longrightarrow \mathbb{C}$$

is differentiable, and by the chain rule, its differential at $t \in]a,b[$ is $\underbrace{J_F(\gamma(t))}_{2\times 2 \text{ matrix}} \cdot \underbrace{J_\gamma(t)}_{2 \text{ columns}}$.

In section 2, we saw that $J_F(\gamma(t))$ is the matrix of multiplication by $F'(\gamma(t)) \in \mathbb{C}$, whereas $J_\gamma(t)$ is the vector $\begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix}$ when we write $\gamma = \gamma_1 + i\gamma_2$.
This corresponds to $\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t)$ and the product is then $F'(\gamma(t)) \cdot \gamma'(t)$, so we do get

$$\int_a^b F'(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b (F\circ\gamma)'(t)\, dt.$$

If we write $F \circ \gamma = u + iv$, then

$$(F\circ\gamma)' = u' + iv'$$

and

$$\int_a^b (F\circ\gamma)'(t)\, dt = \int_a^b u'(t)\, dt + i\int_a^b v'(t)\, dt$$

$$= (u(b) + iv(b)) - (u(a) + iv(a))$$

and this is $F(z_2) - F(z_1)$.

Finally, in the general case, we get

$$\int_a^b f(z)\, dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(z)\, dz$$

$$= (F(a_1) - F(a_0)) + (F(a_2) - F(a_1)) + \cdots$$

$$= F(a_n) - F(a_0).$$

$\square$

<u>Example.</u> Let $U = \mathbb{C}^\times = \mathbb{C} - \{0\}$. Let

$$f : U \longrightarrow \mathbb{C}, \quad z \longmapsto \frac{1}{z}$$

Then f has <u>no</u> primitive.

Indeed, otherwise we would have

$$\int_{\circlearrowleft} f(z)\,dz = 0, \quad \text{but this is}$$

$$\begin{cases} \gamma(t) = e^{it} \\ \gamma'(t) = i e^{it} \end{cases}$$

$$\int_0^{2\pi} f(e^{it})\, i e^{it}\, dt = i \int_0^{2\pi} e^{-it} e^{it}\, dt$$

$$= 2i\pi \neq 0.$$