Chapter III

Cauchy's Theorem

1 – Cauchy's Theorem, Goursat's Theorem, etc...

Cauchy's Theorem is one of the most important properties of holomorphic functions, and we will use it to deduce all of their remarkable pro- -perties.

The "ideal" version of this theorem is :

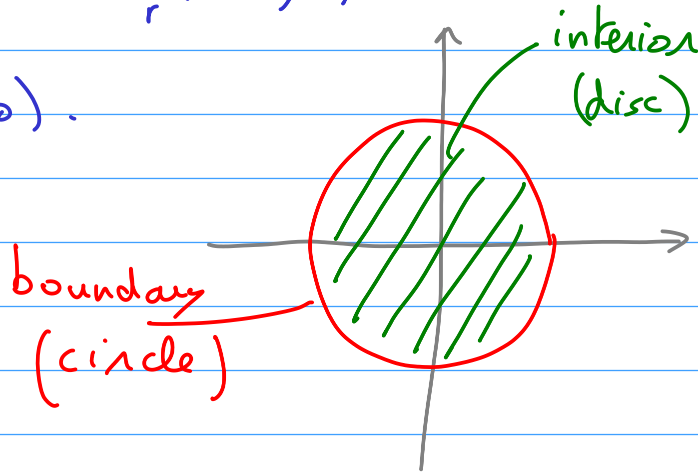
"Theorem" – Let $U \subset \mathbb{C}$ be open, and let $f \in \mathcal{H}(U)$. If γ is a closed curve in U, whose "interior" is contained in U, then

$$\int_\gamma f(z)\,dz = 0 .$$

We have put this in quotes because it is not unambiguous as stated : in the red part, the interior refers to the "inside part" in $\mathbb{C}$

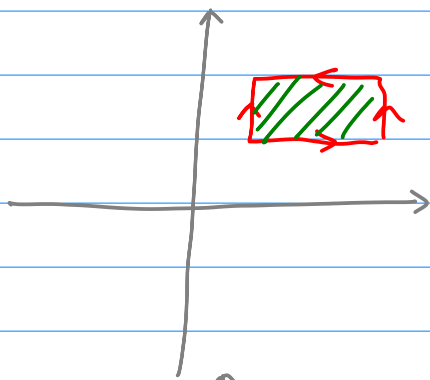
bounded by the curve, and this is far from easy to define rigorously for a general curve γ.

Ex. (1) For γ a circle $C_r(z_0)$, the "inside part" is $\overline{D_r(z_0)}$.

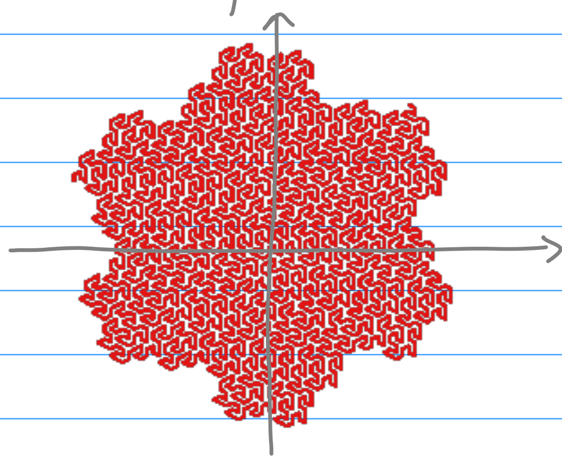


In particular, if γ is the unit circle $C_1(0)$, then we cannot take $U = \mathbb{C}^\times = \mathbb{C} - \{0\}$ in Cauchy's Theorem. This is good since otherwise $f(z) = \frac{1}{z}$, which is in $\mathcal{H}(\mathbb{C}^\times)$, would be a counterexample.

(2) For a square or a rectangle, the interior is also clear.



(3) But what about:

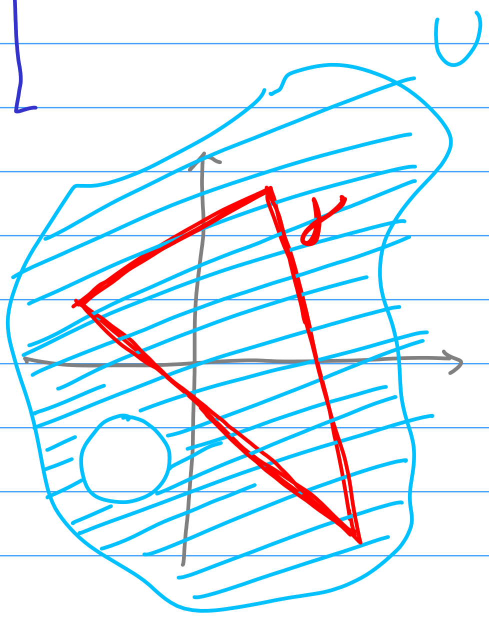


We will work around this difficulty by proving first the result in cases where it is easy to rigorously define the "interior" (in a way that conforms to the geometric intuition).

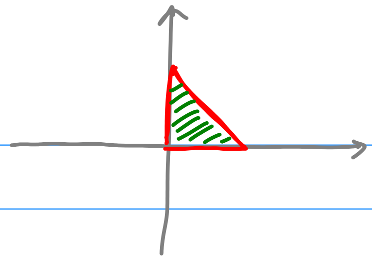
Precisely, we begin with:

Theorem – (Goursat) [Th. II.1.1 of the book]

Let $U \subset \mathbb{C}$ be open, and γ a triangle with interior contained in U. For any $f \in \mathcal{H}(U)$, we have
$$\int_\gamma f(z)\,dz = 0.$$



Triangle: closed curve which is the union of three line-segments = image by an affine linear map $v : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the "standard" triangle
$$T = \left\{ x+iy \;\middle|\; \begin{array}{l} x=0,\ 0 \leq y \leq 1 \\ \text{or } 0 \leq x \leq 1,\ y=0 \\ \text{or } y = 1-x,\ 0 \leq x \leq 1 \end{array} \right\}$$



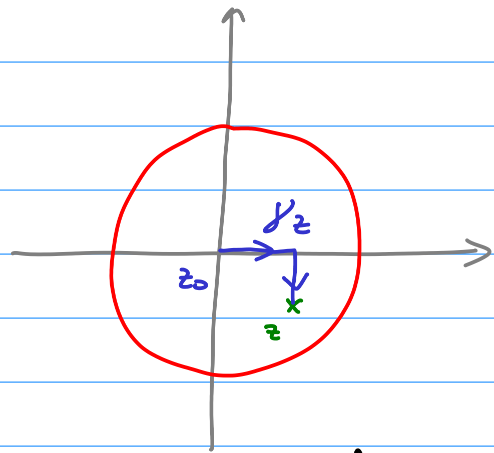
The interior is $u(\tilde{T})$

where $\tilde{T} = \left\{ x+iy \;\middle|\; \begin{array}{l} 0 \le x \le 1, \\ 0 \le y \le 1-x \end{array} \right\}$

We will prove this later, but first deduce some consequences.

Theorem 1 – [II.2.1]

Let $f : D_r(z_0) \longrightarrow \mathbb{C}$ be holomorphic. Then f has a primitive $F \in \mathcal{H}(D_r(z_0))$.



Proof – Guided by the real case, we want to define "$F(z) = \int_{z_0}^{z} f(z)\,dz$" ; to specify what "from z_0 to z" means here, we simply say it means "along the curve γ_z", where γ_z is :

(1) a horizontal segment from z_0 to $z_0 + (x - x_0)$

(2) a vertical segment from $z_0 + (x - x_0)$ to $z_0 + (x - x_0) + i(y - y_0) = z$

So we define $F : D_r(z_0) \longrightarrow \mathbb{C}$ by

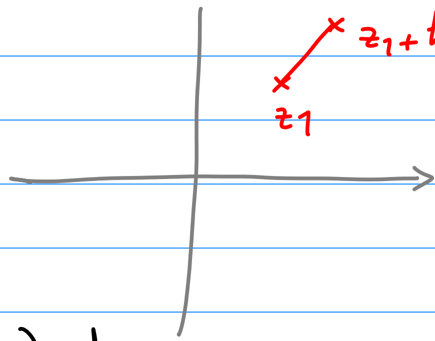
$$F(z) = \int_\gamma f(z)\,dz.$$

We now check that $F \in \mathcal{H}(D_r(z_0))$ is a primitive of f.

Let $z_1 \in D_r(z_0)$ and $h \in \mathbb{C}$ s.t. $z_1 + h \in D_r(z_0)$.

<u>Claim.</u> $F(z_1 + h) - F(z_1) = \int_{\sigma_h} f(z)\,dz$

where σ_h is the line segment from z_1 to $z_1 + h$.



Assuming this, we can write

$$F(z_1+h) - F(z_1) = \int_{\sigma_h} f(z_1)\,dz + \int_{\sigma_h} \big(f(z) - f(z_1)\big)\,dz$$

Now observe that

$$\left| \int_{\sigma_h} \big(f(z) - f(z_1)\big)\,dz \right| \leq \sup_{z \in \sigma_h} |f(z) - f(z_1)| \ \text{length}(\sigma_h)$$

and

$$\int_{\sigma_h} f(z_1)\,dz = f(z_1) \int_{\sigma_h} dz,$$

where (by explicit computation!)

$$\int_{\sigma_h} dz = \text{length}(\sigma_h) = h.$$

Furthermore, since f is continuous (since it is holomorphic) we have

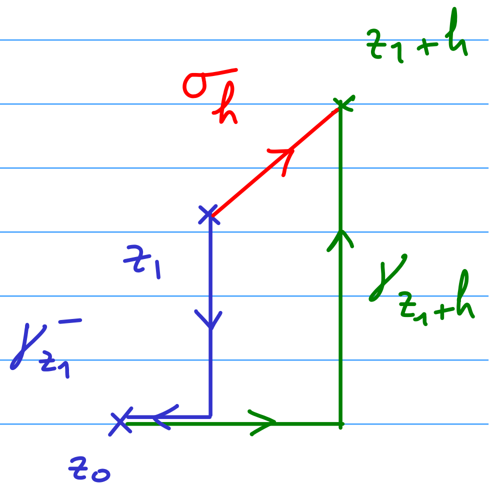
$$\sup_{z \in \sigma_h} |f(z) - f(z_1)| \xrightarrow[h \to 0]{} 0$$

so

$$\left| \frac{F(z_1+h) - F(z_1)}{h} - f(z_1) \right| \leq \sup_{z \in \sigma_h} |f(z) - f(z_1)|$$

implies the result.

Now to check the claim:

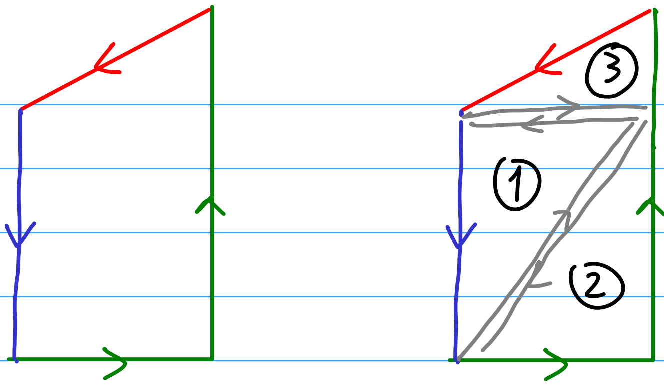


$$F(z_1+h) - F(z_1) = \int f$$

$$= \int f - \int f$$

$$= \int f + \int_{\sigma_h} f .$$

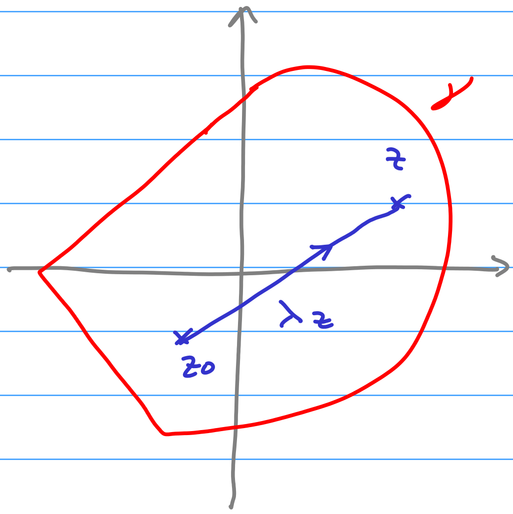
Now the first integral can be expressed as the sum of three "triangular" integrals, so it is zero by Goursat's Theorem:



$$\int f = \int f + \int f + \int f$$

$\square$

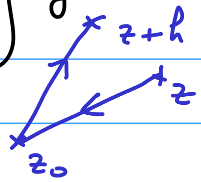
<u>Remark.</u> There is a slightly simpler proof, which also works for $f \in \mathcal{H}(U)$ where U is the "inside" of a <u>convex</u> curve γ (which means that whenever z_1, z_2 are in U, the segment joining them is in U). Namely, we fix $z_0 \in U$, and define a primitive F by putting

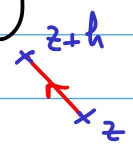


$$F(z) = \int_{\lambda_z} f(w)\,dw$$

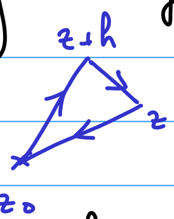
for $z \in U$, where λ_z is the segment joining z_0 to z (oriented from z_0 to z).

Indeed, for h small enough we find that $F(z+h) - F(z) = \int f = \int f$

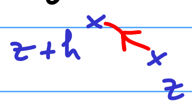




since Goursat's Theorem gives $\int f = 0$. Then we show that



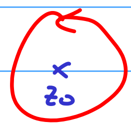
$$\frac{1}{h} \int f \longrightarrow f(z)$$ as in the previous proof.



Now we go further:

Theorem. (Cauchy's Formula) [Th. II.4.1]

Let $U \subset \mathbb{C}$ be open, $\overline{D} = \overline{D_r}(z_0) \subset U$ a disc in U, γ the curve $C_r(z_0)$ with positive orientation.



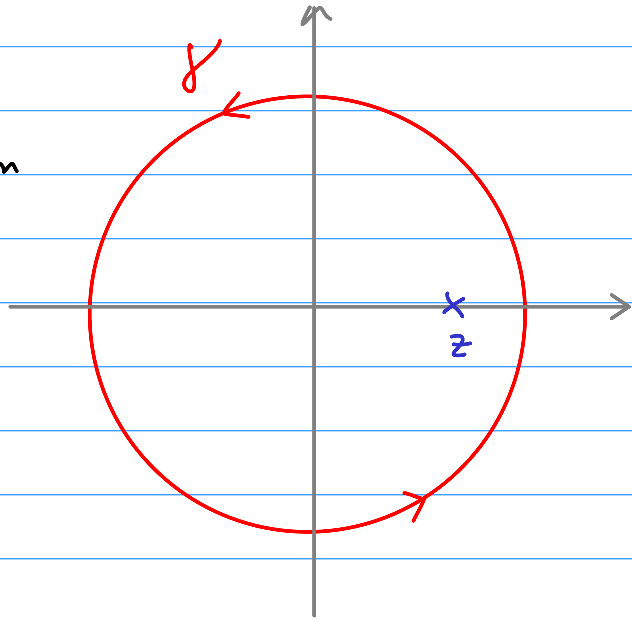
Then for $z \in D_r(z_0)$, we have

$$f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z} \, dw .$$

Proof. Note that the integral on the right-hand side is well-defined, since $w - z \neq 0$ for

$w \in \gamma = C_r(z_0)$.

Up to a rotation and translation, we will assume that $z_0 = 0$ and $z \in \,]-r, r[$ is real.



Note also that the function

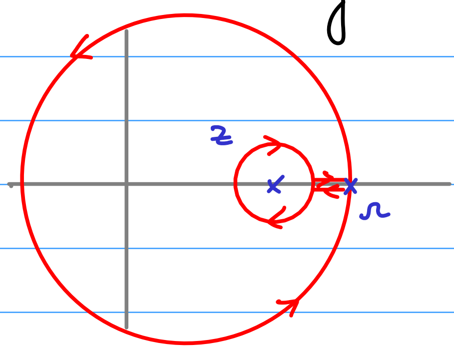
$$g(w) = \frac{f(w)}{w - z}$$

is holomorphic on the set $D_r(0) - \{z\}$,

but not on $D_r(0)$. (In particular, we cannot apply Cauchy's Theorem to say that $\int_\gamma g = 0$.) However for $\varepsilon > 0$ (small enough) we can consider

$$\int_{\gamma_\varepsilon} g(w)\,dw$$

where γ_ε is the following closed curve:



(where the small circle around z has radius ε)

In other words: start from $z+\varepsilon$, go around $C_\varepsilon(z)$ <u>clockwise</u>, then go straight to r, then go around $C_r(0)$ <u>counterclock-wise</u>, then go horizontally (right to left) from r to $z+\varepsilon$.

<u>Claim</u>: $$\int_{\gamma_\varepsilon} g(w)\,dw = 0.$$

Assuming this, note that this means

$$-\int_{C_\varepsilon(z)} g(w)\,dw + \int_{C_r(z)} g(w)\,dw = 0$$

so we are reduced to computing

$$\int_{C_\varepsilon(z)} g(w)\,dw$$

(which we will do by letting $\varepsilon \to 0$...)

Indeed, note that

$$g(w) = \frac{f(w)}{w-z} = \frac{f(z)}{w-z} + \frac{f(w)-f(z)}{w-z}$$

for $w \in C_\varepsilon(z)$. Then

$$\int_{C_\varepsilon(z)} \frac{f(z)}{w-z}\,dw = f(z)\int_{C_\varepsilon(z)} \frac{dw}{w-z}$$

which is equal to $2i\pi f(z)$ (using the parameteri-
-zation $\begin{cases} [0,2\pi] \longrightarrow \mathbb{C} \\ t \longmapsto z + \varepsilon e^{it} \end{cases}$, we get $\displaystyle\int_{C_\varepsilon(z)} \frac{dw}{w-z} = \varepsilon \int_0^{2\pi} e^{it} e^{-it} \, dt$).

On the other hand:

$$\left| \int_{C_\varepsilon(z)} \frac{f(w) - f(z)}{w - z} \, dw \right| \leq 2\pi\varepsilon \sup_{w \in C_\varepsilon(z)} \left| \frac{f(w) - f(z)}{w - z} \right|$$

length of $C_\varepsilon(z)$

and since $|w - z| = \varepsilon$ and the limit

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} \quad \text{exists}$$

the ratio is bounded as $\varepsilon \longrightarrow 0$. We conclude that

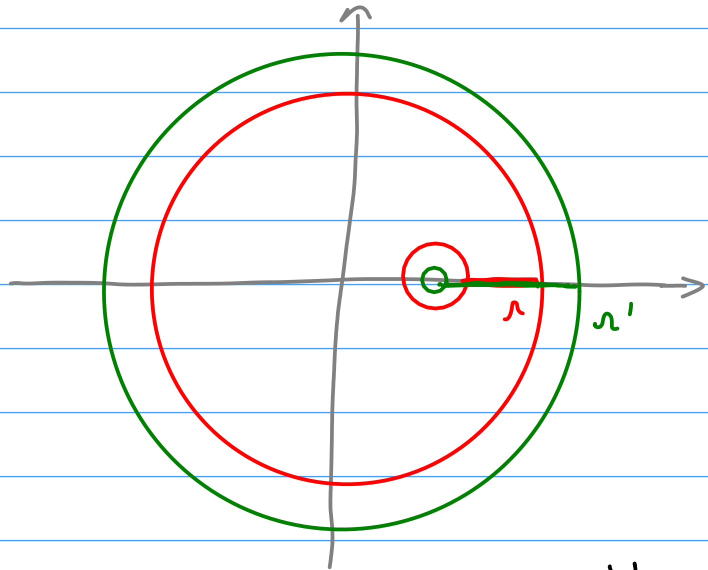
$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} g(w) \, dw = 2i\pi f(z)$$

and hence the result.

Now for the <u>claim</u>: it will follow from the fact that g has a primitive in an open set V_ε such that γ_ε is almost a closed curve in V_ε. The following choice will work.

First, we observe that since $\overline{D}_r(z_0) \subset U$, we can "enlarge" it a bit: there exists $r' > r$ such that $D_{r'}(z_0) \subset U$. Then we define

$$V_\varepsilon = \left\{ w \in D_{r'}(z_0) \;\middle|\; |w - z| > \frac{\varepsilon}{2}, \; w \notin \left[z + \frac{\varepsilon}{2}, r\right] \right\}$$

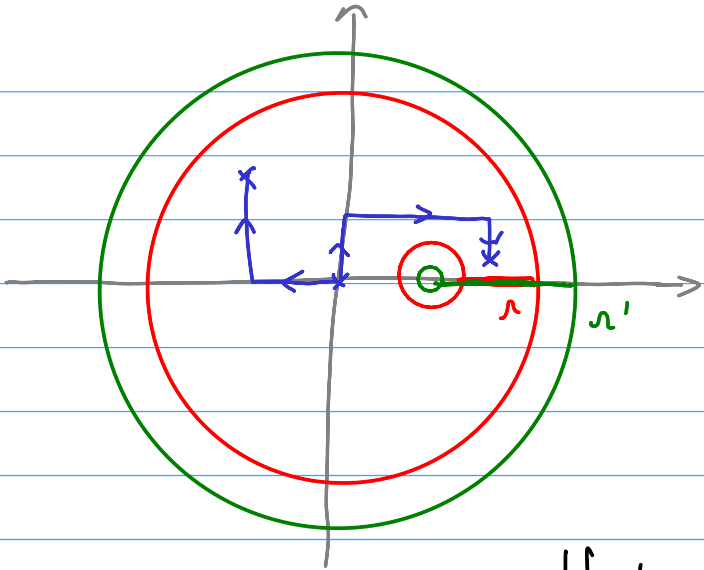


Note that $V_\varepsilon \subset U$ is open, but γ_ε is <u>not</u> a curve in V_ε (the horizontal segments are not contained in V_ε).

Still, we can first check that g has a primitive G in V_ε by defining

$$G(w) = \int_{\sigma_w} g(\lambda)\, d\lambda$$

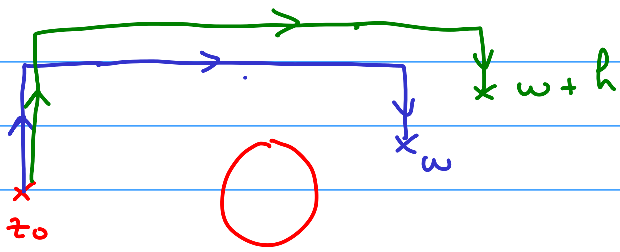
where σ_w is a fixed polygonal path joining z_0 to w (which clearly exist and can be described explicitly).



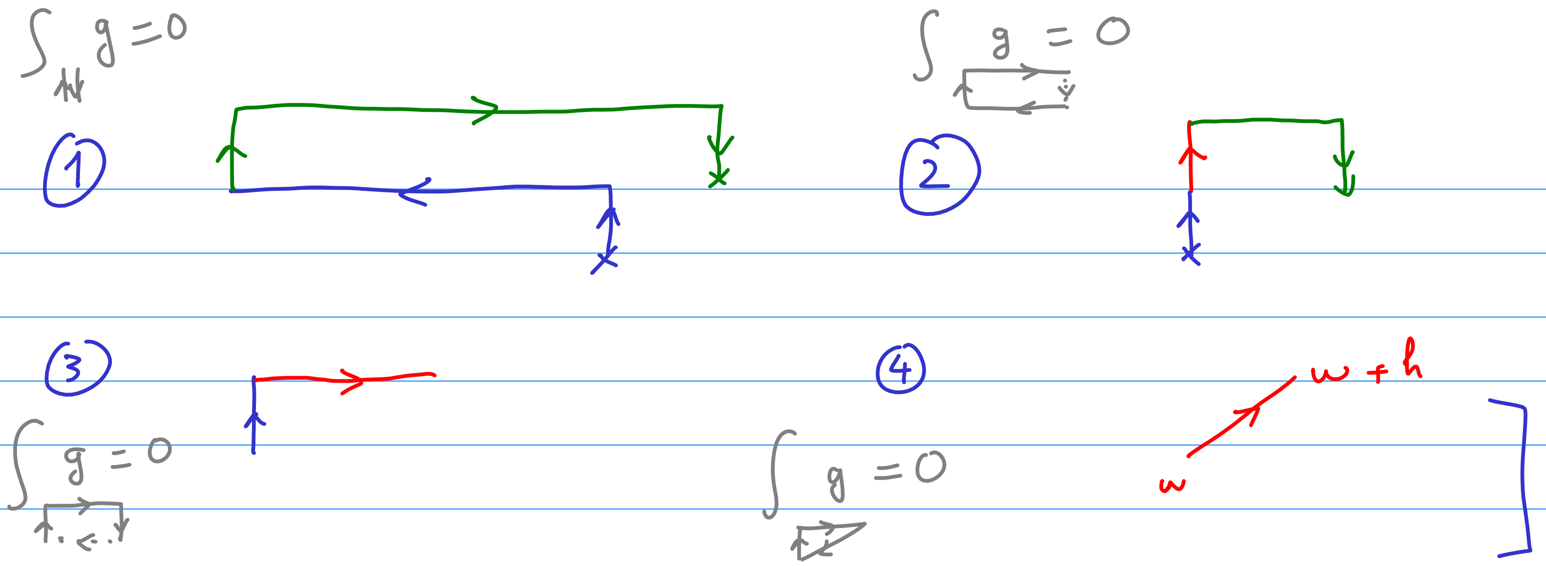
The fact that G is a primitive of g is proved as before by observing that $G(w+h) - G(w) = \int g$

(for h small enough); to check this, it is useful to observe that we can use Theorem 1 to deduce that $\int_\alpha g = 0$ if α is a closed curve in a disc (or even a rectangle, etc) contained in V_ε.

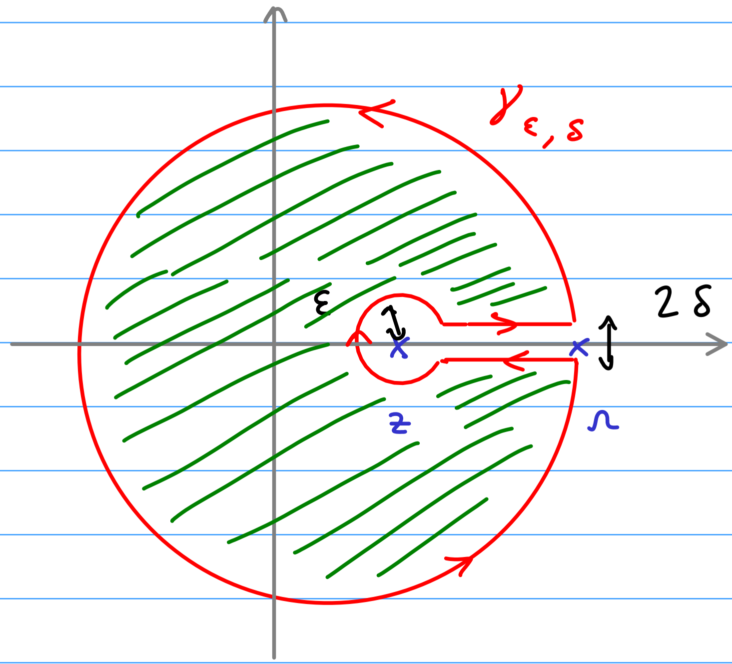
[For instance: consider



The difference of the green minus blue line integrals is equal successively to the integrals along:



Once we have the primitive G, we consider some new curves $\gamma_{\varepsilon,\delta}$, for $\delta > 0$ small enough, which <u>are</u> in V_ε:



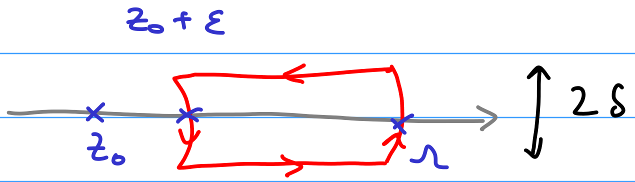
Since g has a primitive in V_ε, we know that

$$\int_{\gamma_{\varepsilon,\delta}} g = 0.$$

Then the last step is to show that

$$\lim_{\delta \to 0} \int_{\gamma_{\varepsilon\delta}} g(\lambda)\, d\lambda = \int_{\gamma} g(\lambda)\, d\lambda .$$

This final property will be deduced from the continuity of g : the difference of integrals is the integral along the curve below :



Indeed, g is continuous, hence bounded, inside this curve, which already shows that the integrals over the two short arcs)) tend to zero as $\delta \to 0$ (because the length of these arcs is proportional to δ). Moreover, the uniform continuity of g inside the curve implies that $|g(x+i\delta) - g(x-i\delta)|$ becomes arbitrarily small for all x such that $x \pm i\delta$ is in the horizontal part, provided δ is small enough. This means that

$$\int_{\substack{\leftarrow \\ \rightarrow}} g(w)\, dw = \int \big(g(x+i\delta) - g(x-i\delta)\big)\, dx$$

tends to zero as $\delta \longrightarrow 0$.

$\square$

<u>Remark</u> - We can alredy see some remarkable consequences of Cauchy's Integral Formula.

For instance, observe that from

$$f(z) = \int_\gamma \frac{f(w)}{w-z}\, dw$$

for <u>all</u> $z \in D_r(z_0)$, with γ <u>independent</u> of z, it follows immediately that the values of f in the whole (2-dimensional) disc are determined by those on the circle γ (which is 1-dimensional). In particular, if f is zero on the circle, then it is zero on the whole disc!

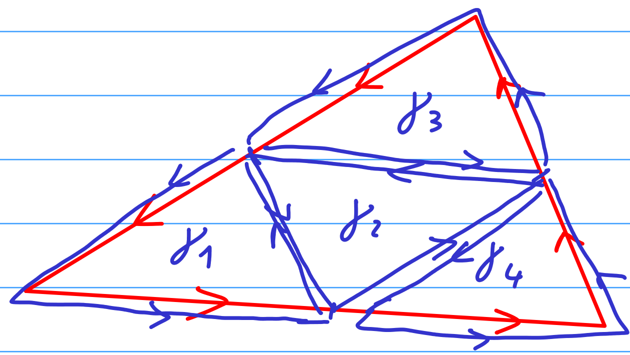
2 - Proof of Goursat's Theorem

Recall the statement: if $f \in \mathcal{H}(U)$ and γ is a triangle whose interior is contained in U, then

$$\int_\gamma f(z)\,dz = 0.$$

The key lemma is the following:

Lemma - Let γ be a triangle with interior in U. Let $\gamma_1, \dots, \gamma_4$ be the triangles in the following picture:



There exists $j \in \{1,2,3,4\}$ such that

$$\left| \int_\gamma f(z)\,dz \right| \leq 4 \left| \int_{\gamma_j} f(z)\,dz \right|$$

Proof - Looking at the orientation and segments in opposite directions that cancel out, we have

$$\int_\gamma f = \sum_{j=1}^{4} \int_{\gamma_j} f.$$

Hence

$$\left| \int_\gamma f \right| \leq 4 \max_{1 \leq j \leq 4} \left| \int_{\gamma_j} f \right| ,$$

and the maximum is one of the four integrals.

$\square$

We can iterate this lemma, since the interiors of the four triangles are also in U.

Notice that the length of γ_j is $\frac{1}{2}$ length(γ) (because each γ_j is similar to the image of γ by $z \longmapsto \frac{1}{2} z$) and the diameter is also half of that of γ.

So iterating, we find for any $n \geq 1$ some triangle $\gamma^{(n)}$ in U, of diameter 2^{-n} diam(γ) and length 2^{-n} length(γ), such that

$$\left| \int_\gamma f \right| \leq 4^n \left| \int_{\gamma^{(n)}} f \right| .$$

Moreover, the interior of $\gamma^{(n+1)}$ is always contained in that of $\gamma^{(n)}$.

Fact: let $T^{(n)}$ be the filled triangle bounded by and including $\gamma^{(n)}$. There exists $z_0 \in \bigcap_{n \geq 1} T^{(n)}$.

Assume that this is true; then we use the fact that f is holomorphic at z_0:

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0) g(z)$$

where $\lim_{z \to z_0} g(z) = 0$. So

$$\int_{\gamma^{(n)}} f(z)\, dz = \underbrace{\int_{\gamma^{(n)}} f(z_0)\, dz}_{=0}$$

$$+ \underbrace{f'(z_0) \int_{\gamma^{(n)}} (z - z_0)\, dz}_{=0}$$

$=0$ because $\frac{1}{2}(z - z_0)^2$ is a primitive

$$+ \int_{\gamma^{(n)}} (z - z_0)\, g(z)\, dz.$$

This gives

$$\left| \int_{\gamma^{(n)}} f(z)\, dz \right| \leq \sup_{z \in \gamma^{(n)}} |z - z_0| \sup_{z \in \gamma^{(n)}} |g(z)| \times \text{length}(\gamma^{(n)})$$

$\leq \text{diam}(\gamma^{(n)})$ since z and z_0 are in the "closed" triangle

$$\leq 4^{-n}\, \text{diam}(\gamma)\, \text{length}(\gamma) \times \sup_{z \in \gamma^{(n)}} |g(z)|$$

and therefore

$$\left| \int_\gamma f(z)dz \right| \leq \text{diam}(\gamma)\ \text{length}(\gamma) \sup_{z \in \gamma^{(n)}} |g(z)|$$

which tends to 0 as $n \to \infty$, hence the result.

There remains to prove the fact above. This is a general property of compact (closed, bounded) sets.

Precisely let z_n be any element of the "closed" triangle $T^{(n)}$ (for instance the center of mass).

Let $n \geq m \geq 1$ be integers. Then observe that

$|z_n - z_m| \leq \text{diam}(\gamma^{(m)})$ (both points are in the closed triangle) so that (z_n) is a Cauchy sequence. Let z_0 be its limit. Then for $m \geq 1$, note that z_n being in the closed m-th triangle for $n \geq m$ implies that z_0 is also. Hence z_0 belongs to all triangles.

$\square$

Remark. Where did we use the fact that f is holomorphic and not only differentiable as a function of two real variables?

It is simply when saying that

$$\int_\gamma f'(z_0)(z - z_0)\, dz = 0$$

because there is a primitive of $(z - z_0)$. This depends essentially on the fact that the differential at z_0 is multiplication by $f'(z_0)$, and not a general linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.