Chapter IV

Applications of Cauchy's Theorem

Even in the partial form we have obtained, Cauchy's Theorem (and the integral formula) are extraordinarily powerful. We will use them to prove the key fundamental (extraordinary) properties of holomorphic functions.

1. Analyticity

We begin with maybe the most striking fact:
[Cor. II.4.2, Th. II.4.4]

Theorem. $U \subset \mathbb{C}$ open ; $f \in \mathcal{H}(U)$

(1) The derivative f' is also in $\mathcal{H}(U)$ [so f is indefinitely differentiable !]

(2) Let $z_0 \in U$ and $r > 0$ such that $D_r(z_0)$ is contained in U. Then the Taylor series of f at z_0 has radius of convergence $\geq r$ and its sum is equal to f on $D_r(z_0)$:

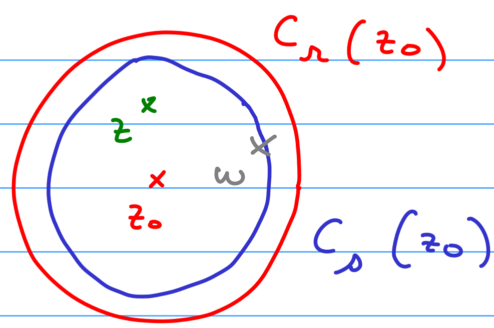
$$\forall z \in D_r(z_0), \qquad f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$$
where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$.

This shows that, at least locally, holomorphic functions are "the same" as convergent power series. Note however that the Taylor series at z_0 may not converge on all of U.

Proof. We actually start with (2). Let z_0, r satisfy $D_r(z_0) \subset U$. Fix $s \in]0, r[$ and let γ be the circle of radius s centered at z_0. Then γ is contained in U and f is holomorphic in $D_s(z_0)$. By Cauchy's Integral formula, we get

$$f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z}\, dw$$

for all $z \in D_s(z_0)$.



But now we write

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)}$$

$$= \frac{1}{w-z_0} \frac{1}{1-\frac{z-z_0}{w-z_0}}$$

$$= \frac{1}{w-z_0} \sum_{n=0}^{+\infty} \left(\frac{z-z_0}{w-z_0}\right)^n$$

for $w \in \gamma$ since then

$$\left|\frac{z-z_0}{w-z_0}\right| = \frac{|z-z_0|}{s} < 1.$$

Moreover, the convergence is uniform for $w \in \gamma$ (but z fixed here) since the upper-bound $\frac{|z-z_0|}{s}$ is independent of $w \in \gamma$. So we can exchange the series and the integral:

$$f(z) = \frac{1}{2i\pi} \int_\gamma f(w) \sum_{j=0}^{+\infty} (z-z_0)^n (w-z_0)^{-n-1} \, dw$$

$$= \sum_{n=0}^{+\infty} a_n (z-z_0)^n$$

for all $z \in D_s(z_0)$, where

$$a_n = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}} \, dw.$$

[So, essentially, we are saying that

(1) by Cauchy's Formula, f is an "average" of functions $g_w(z) = \dfrac{f(w)}{w-z}$ which each have power series expansions

(2) "averaging" such functions preserves the existence of power series.]

So f is the sum of the power series

$$\sum_{n=0}^{+\infty} a_n (z-z_0)^n$$

in $D_\delta(z_0)$. We know that this implies that

$$f'(z) = \sum_{n=1}^{+\infty} n a_n (z-z_0)^{n-1}$$
$$= \sum_{n=0}^{+\infty} (n+1)\, a_{n+1} (z-z_0)^n$$

which is also a power series, so holomorphic in $D_\delta(z_0)$. Inductively, all derivatives of f exist in $D_\delta(z_0)$ and evaluating at z_0 gives

$$a_0 = f(z_0) \qquad \text{(Cauchy's Formula)}$$

$$1 \cdot a_1 = f'(z_0)$$

$$\vdots$$

$$n!\, a_n = f^{(n)}(z_0).$$

Finally, since the power series has coefficients independent of s and converges for $z \in D_s(z_0)$ for <u>all</u> $0 < s < r$, its radius of convergence must be $\geq r$. This finishes the proof of (2).

We deduce (1) simply from the fact that for every $z_0 \in U$, there exists $r > 0$ such that the disc $D_r(z_0)$ is contained in U, and the fact that being holomorphic is a <u>local</u> property: since (by (2)) f is the sum of its Taylor series in $D_r(z_0)$, it's derivative is also a power series in this disc, so is also holomorphic.

$\square$

<u>Corollary</u> (of the proof). If $f \in \mathcal{H}(U)$ and $z_0 \in U$, then for any $r > 0$ such that $\overline{D}_r(z_0) \subset U$, we have

$$f^{(n)}(z_0) = \frac{n!}{2i\pi} \int_{\gamma_r} \frac{f(w)}{(w - z_0)^{n+1}} \, dw$$

where γ_r is the circle $C_r(z_0)$ oriented counter-clockwise.

In particular, we have

$$|f^{(n)}(z_0)| \leq n! \frac{\sup_{|w-z_0|=r} |f(w)|}{r^n}$$

(Cauchy's Inequalities)

Proof. We saw that

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-z_0)^n$$

with $a_n = \frac{1}{2i\pi} \int_{\gamma_r} \frac{f(w)}{(w-z_0)^{n+1}} dw$

and we know also that for such a power series, we have $a_n = \frac{f^{(n)}(z_0)}{n!}$.

We obtain Cauchy's Inequalities by observing that

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \times \underbrace{\text{length}(\gamma_r)}_{=2\pi r} \times \underbrace{\sup_{w\in\gamma_r} \frac{|f(w)|}{|w-z_0|^{n+1}}}_{=\sup_{w\in\gamma_r} \frac{|f(w)|}{r^{n+1}}}$$

$$= \frac{n!}{r^n} \sup_{|w-z_0|=r} |f(w)|.$$

$\square$

2. Liouville's Theorem

Theorem (Liouville). If $f \in \mathcal{H}(\mathbb{C})$ and f is bounded ($\exists M \geq 0, \forall z \in \mathbb{C}, |f(z)| \leq M$) then f is constant.

Proof. Let $z_0 = 0$; since $D_r(0) \subset \mathbb{C}$ for all $r > 0$, we have the power series expansion

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!}$$

for all $z \in \mathbb{C}$ (picking r such that $|z| < r$; in other words, the radius of convergence is $+\infty$).

Let $n \geq 1$ and $r > 0$. Cauchy's Inequalities give

$$|a_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{1}{r^n} \sup_{|w|=r} |f(w)|$$

$$\leq \frac{M}{r^n}$$

which means that $a_n = 0$, since r is arbitrarily large. Hence $f(z) = a_0$ for all $z \in \mathbb{C}$.

$\square$

Corollary (Fundamental th. of algebra) – Let f be a polynomial with complex coefficients of degree $d \geq 1$. Then f has a complex root (i.e. $\exists z_0$ s.t. $f(z_0) = 0$.)

Proof – We assume the opposite. The function g such that $g(z) = \frac{1}{f(z)}$ is then in $\mathcal{H}(\mathbb{C})$.

We claim that g is bounded, in which case it would be a constant, but then so would be f, contradicting the fact that $\deg(f) \geq 1$.

To check the claim, write $d = \deg(f) \geq 1$ and

$$f(z) = a_d z^d + \dots + a_1 z + a_0$$

where $a_d \neq 0$, $a_i \in \mathbb{C}$.

For $z \in \mathbb{C}$, we have

$$|f(z)| \geq |a_d| \, |z|^d - \sum_{i=0}^{d-1} |a_i| \, |z|^i.$$

and for $|z| \geq 1$, this gives

$$|f(z)| \geq |a_d| \, |z|^d - C \, |z|^{d-1}$$

where $C = \sum_{i=0}^{d-1} |a_i|$. So, still for $|z| \geq 1$, we get $$|f(z)| \geqslant |a_d|\,|z|^{d-1}\left(|z| - \frac{C}{|a_d|}\right).$$

Suppose then that
$$|z| \geqslant \frac{C}{|a_d|} + 1.$$

We get $$|f(z)| \geqslant |a_d|\left(1 + \frac{C}{|a_d|}\right)^{d-1}.$$

So for $|z| \geqslant 1 + \frac{C}{|a_d|}$, we have
$$|g(z)| \leqslant \frac{1}{D}, \qquad D = |a_d|\left(1 + \frac{C}{|a_d|}\right)^{d-1}.$$

and therefore, for $z \in \mathbb{C}$ arbitrary
$$|g(z)| \leqslant \max\left(\sup_{|z| \leq C+1} |g(z)|,\ D\right)$$

and so g is bounded (since it is continuous for $|z| \leq C+1$, so $\sup\limits_{|z| \leq C+1} |g(z)|$ exists in $\mathbb{R}$), as desired.

$\square$

<u>Remark.</u> This result is special to functions holomorphic on <u>all of $\mathbb{C}$</u>! For instance, let
$$U = \{ z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \}$$
and $f(z) = \dfrac{1}{z+1}$. Then $f \in \mathcal{H}(U)$ and

f is bounded, since $|f(z)| = \frac{1}{|z+1|} \leq 1$ if $\mathrm{Re}(z) > 0$.

3. Zeros of holomorphic functions

If f is a non-zero polynomial, it has only finitely many zeros. The same is not true of all holomorphic functions (ex. if we define $f(z) = \frac{e^{iz} - e^{-iz}}{2i} = \sin(z)$, then $f(2k\pi) = 0$ for all $k \in \mathbb{Z}$) but an important weaker pro-perty holds: the zeros are "isolated" (if they exist: one can have $f \in \mathcal{H}(\mathbb{C})$ without zeros, for instance $f(z) = e^z$).

Definition (Order of vanishing) Let $U \subset \mathbb{C}$ be open, let $f \in \mathcal{H}(U)$ and $z_0 \in U$. The order of vanishing of f at z_0, denoted $\mathrm{ord}_{z_0}(f)$, is either $+\infty$ if $f^{(k)}(z_0) = 0$ for all $k \geq 0$, or the integer $k \geq 0$ such that $f(z_0) = \dots = f^{(k-1)}(z_0) = 0$, $f^{(k)}(z_0) \neq 0$.

(If $f(z_0) \neq 0$, then $k = 0$.)

Proposition. Let U, z_0 be as in the definition. Let $f \in \mathcal{H}(U)$

(1) If $\mathrm{ord}_{z_0}(f) = +\infty$, then $f(z) = 0$ for any $z \in D_r(z_0)$ such that $D_r(z_0) \subset U$. (So f is then locally zero.)

(2) Otherwise, there exists $\tilde{f} \in \mathcal{H}(U)$ such that $f(z) = (z - z_0)^{\mathrm{ord}_{z_0}(f)} \tilde{f}(z)$ for all $z \in D_r(z_0)$
$\tilde{f}(z_0) \neq 0$.

(3) For any $g \in \mathcal{H}(U)$, we have

$$\mathrm{ord}_{z_0}(f + g) \geq \min(\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g))$$

$$\mathrm{ord}_{z_0}(fg) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g).$$

Proof. (1) For $z \in D_r(z_0)$, we have

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n = 0$$

if $\mathrm{ord}_{z_0}(f) = +\infty$.

(2) By definition of $k = \mathrm{ord}_{z_0}(f)$, if it is not $+\infty$, we can write

$$f(z) = \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \sum_{n=k+1}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for z close to z_0. This means that

$$f(z) = (z - z_0)^k \, \tilde{f}(z)$$

where $\tilde{f}(z) = \displaystyle\sum_{n=0}^{+\infty} \frac{f^{(n+k)}(z_0)}{(n+k)!}(z - z_0)^n$.

Then $\tilde{f} \in \mathcal{H}(D_r(z_0))$ and $\tilde{f}(z_0) = \dfrac{f^{(k)}(z_0)}{k!}$ is non-zero.

(3) Since $(f+g)^{(j)}(z_0) = f^{(j)}(z_0) + g^{(j)}(z_0)$ we have $(f+g)^{(j)}(z_0) = 0$ if $f^{(j)}(z_0) = g^{(j)}(z_0) = 0$ which means $\mathrm{ord}_{z_0}(f+g) \geq \min(f^{(j)}(z_0), g^{(j)}(z_0))$.

And if we write

$$\begin{cases} f(z) = (z - z_0)^{\mathrm{ord}_{z_0}(f)} \, \tilde{f}(z) \\ g(z) = (z - z_0)^{\mathrm{ord}_{z_0}(g)} \, \tilde{g}(z) \end{cases}$$

for $z \in D_r(z_0)$ with $\tilde{f}(z) \neq 0$, $\tilde{g}(z) \neq 0$, then

$$(fg)(z) = (z - z_0)^{\mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)} \, (\tilde{f}\tilde{g})(z)$$

with $(\tilde{f}\tilde{g})(z_0) \neq 0$. Comparing with the power series expansion of fg, this means that $\mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g) = \mathrm{ord}_{z_0}(fg)$.

[A different argument: by the Leibniz formula

$$(fg)^{(j)}(z_0) = \sum_{k=0}^{j} \binom{j}{k} f^{(k)}(z_0)\, g^{(j-k)}(z_0);$$

if $0 \leq j < \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$, then if $0 \leq k \leq j$, <u>either</u> $k < \mathrm{ord}_{z_0}(f)$ or $j-k < \mathrm{ord}_{z_0}(g)$ so the sum is $0 + 0 + \dots = 0$. On the other hand, for $j = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$, the only possibly non-zero term is $\begin{cases} k = \mathrm{ord}_{z_0}(f) \\ j-k = \mathrm{ord}_{z_0}(g) \end{cases}$, which gives

$$(fg)^{j}(z_0) = \binom{j}{k} f^{(k)}(z_0)\, g^{(j-k)}(z_0) \neq 0,$$

which shows that $\mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g) = \mathrm{ord}_{z_0}(fg)$.]

$\square$

This proposition gives the "local" picture. Here is the global statement.

Corollary -

Let $U \subset \mathbb{C}$ be open, $z_0 \in U$ and $f \in \mathcal{H}(U)$. Assume $f(z_0) = 0$ (i.e. $\mathrm{ord}_{z_0}(f) \geq 1$).

If $\mathrm{ord}_{z_0}(f) \neq +\infty$ then there exists $\delta > 0$ such that $f(z) \neq 0$ if $z \neq z_0$ and $|z - z_0| < \delta$.

(" z_0 is an isolated zero ").

Proof - Write $f(z) = (z - z_0)^{\mathrm{ord}_{z_0}(f)} \tilde{f}(z)$ for $z \in D_r(z_0)$. Then, for $z_0 \neq z \in D_r(z_0)$, we have $f(z) = 0 \iff \tilde{f}(z) = 0$. But $\tilde{f}(z_0) \neq 0$ and $\tilde{f}$ is continuous, so there exists $0 < \delta \leq r$ such that $\tilde{f}(z) \neq 0$ for $|z - z_0| < \delta$.

$\square$

4 - The principle of analytic continuation

Theorem - [II . 4 . 8] Let $U \subset \mathbb{C}$ be open and connected. Let $f \in \mathcal{H}(U)$. Let $Z \subset U$ be an infinite set with a limit point $z_0 \in U$, $z_0 \notin Z$. If $f(z) = 0$ for all $z \in Z$, then $f = 0$.

Proof. Since f is continuous and zero on Z, it is also zero on any limit point, in particular $f(z_0) = 0$. Then, since any disc $D_\varepsilon(z_0)$ with $\varepsilon > 0$ contains some element $z \in Z$ (by definition of limit points), and $z \neq z_0$, the zero z_0 of f is not isolated, which implies that $\mathrm{ord}_{z_0}(f) = +\infty$, so (by the results of the previous section), the function f is zero on any disc $D_r(z_0)$ contained in U.

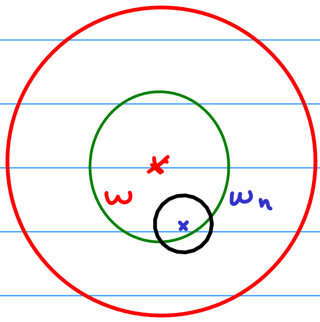
Up to now, we have not used the assumption that U is connected. But define

$$\varphi : U \longrightarrow \{0, 1\}$$

by $\varphi(z) = 1$ if and only if f is zero on some open disc $D_r(z)$, $r > 0$, around z.

The function φ is <u>continuous</u> : let w_n in U converge to $w \in U$, and let $\alpha = \varphi(w)$.

(1) If $\varphi(w) = 1$, then f is zero in some disc $\left\{ D_r(w) \atop r>0 \right.$; for all $n \geqslant n_0$ we have $w_n \in D_{r/2}(w)$ (definition of $w_n \longrightarrow w$) so f is zero in $D_{r/4}(w_n)$ (because $D_{r/4}(w_n) \subset D_r(w)$) so $\varphi(w_n) = 1$



for all n large enough,

so $\varphi(w_n) \longrightarrow \varphi(w)$.

(2) If $\varphi(w) = 0$, then the order of vanishing of f at w is not $+\infty$, so w is the only possible zero of f in some $D_\delta(w)$, $\delta > 0$. All w_n are in $D_{\delta/2}(w)$ for n large enough, and then $D_{\delta/4}(w_n)$ contains at most one zero of f, which means $\varphi(w_n) = 0$. So $\varphi(w) = 0 = \lim\limits_{n \to \infty} \varphi(w_n)$.

Now since $\varphi : U \longrightarrow \{0, 1\}$ is continuous and

U is connected, we know that φ is constant. But, by the beginning of the proof, we know that $\varphi(z_0) = 1$, so $\varphi = 1$ everywhere, and in particular this means that $f = 0$ everywhere (because $\varphi(w) = 1 \Rightarrow f(w) = 0$ by definition).

$\square$

Corollary. If $U \subset \mathbb{C}$ is connected and open, and if $f, g \in \mathcal{H}(U)$ coincide on a set $Z \subset U$ with a limit point $z \in U$, $z \notin Z$, then $f = g$. In particular, if $f = g$ on an open subset $\emptyset \neq V \subset U$, or on a segment $\gamma \subset U$, then $f = g$.

Proof. Apply the previous result to $f - g$. For the special cases, note that if $V \subset U$ is open and $V \neq \emptyset$, then $D_r(z_0) \subset V$ for some $z_0 \in V$, $r > 0$, and then

$$Z = \left\{ z_0 + \frac{r}{n+1} \right\} \subset V$$

has the limit point $z_0 \in U - Z$.

Similarly for a segment.

$\square$

<u>Remark.</u> The condition that the limit point is in U is essential: take for instance $U = \mathbb{C} - \{0\}$ and $f(z) = \sin(1/z) = \dfrac{e^{i/z} - e^{-i/z}}{2i}$.

Then $f \in \mathcal{H}(U)$ and $f \not\equiv 0$, but

$$f\left(\frac{1}{2k\pi}\right) = 0$$

for any integer $k \geq 1$, with $\frac{1}{2k\pi} \xrightarrow[k\to\infty]{} 0$.

The result implies the following principle, called "Principle of analytic continuation": if $f \in \mathcal{H}(U)$ and $V \supset U$ is open and connected, then there is <u>at most one</u> $\tilde{f} \in \mathcal{H}(V)$ such that $\tilde{f}(z) = f(z)$ for all $z \in U$. When $\tilde{f}$ exists, one says that "f has analytic continuation to V".

5. Limits of holomorphic functions

Theorem 1. [II.5.2] Let $U \subset \mathbb{C}$ be open and let $(f_n)_{n \geq 1}$ be a sequence of holomorphic functions on U. Suppose that $f_n(z) \longrightarrow f(z)$ for all $z \in U$, uniformly on compact subsets. Then the limit f is in $\mathcal{H}(U)$.

Remarks. (1) This is again very striking! For functions which are only differentiable, the assumption would only imply that f is continuous!

(2) We have implicitly seen two examples of this principle already: power series (limits of polynomials) and functions like $\frac{1}{2i\pi}\int_\gamma \frac{f(w)}{w-z}\,dw$ (the integral being also a limit of some kind).

For the proof, we need a lemma which is useful on its own:

Lemma ("Morera's Theorem"; II.5.1). Let $U \subset \mathbb{C}$

be open and let $f : U \longrightarrow \mathbb{C}$ be a continuous function. Assume that for any open disc $D \subset U$ and any triangle γ with "inside" contained in D, we have $\int_\gamma f(z)\,dz = 0$. Then $f \in \mathcal{H}(U)$.

(So this is a converse to Goursat's Theorem.)

<u>Proof.</u> Let $D_r(z_0) \subset U$ be an open disc. Define

$$F(z) = \int_{\gamma_z} f(z)\,dz$$

for $z \in D_r(z_0)$, where γ_z is the line segment joining z_0 to z. Arguing as in Theorem 1 of Chapter <u>III</u>, we get

$$F(z+h) - F(z) = \int_{[z,\, z+h]} f(w)\,dw$$

and then

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

using the fact that f is continuous at z.

So $F \in \mathcal{H}(D_r(z_0))$ is holomorphic; since $F' = f$ on $D_r(z_0)$, and F' is holomorphic it follows that $f \in \mathcal{H}(D_r(z_0))$ also, and then $f \in \mathcal{H}(U)$ since z_0 is arbitrary.

$\square$

<u>Proof of Th. 1</u> – Let $D_r(z_0) \subset U$ be an open disc in U, and γ a triangle with inside contained in $D_r(z_0)$. By Goursat's Theorem, we have

$$\int_\gamma f_n(z)\,dz = 0 \quad \text{for } n \geq 1.$$

Since $f_n(z) \longrightarrow f(z)$ on compact subsets, the limit f is continuous.

Since γ is compact in U, $f_n(z) \longrightarrow f(z)$ uniformly for all $z \in \gamma$, and so

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0.$$

By Morera's Theorem, we conclude that f is holomorphic.

$\square$

[Th. II.5.3]

<u>Corollary</u> – Under the assumptions of the Theorem,

we also have $f_n'(z) \longrightarrow f'(z)$ uniformly over any compact set.

Proof. It suffices to prove that if $\overline{D}_r(z_0) \subset U$, then the convergence holds and is uniform on $\overline{D}_r(z_0)$. (Because any compact set will be contained in the union of finitely many such discs.) To see this, pick $s > r$ such that $D_s(z_0) \subset U$ and let $\rho = \frac{r+s}{2} \in]r, s[$, and $\gamma = C_\rho(z_0)$.

Then $\overline{D}_r(z_0) \subset D_\rho(z_0)$ and we have

$$f'(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{(w-z)^2} dw$$

$$f_n'(z) = \frac{1}{2i\pi} \int_\gamma \frac{f_n(w)}{(w-z)^2} dw$$

by the Cauchy formula for f'.

For $z \in \overline{D}_r(z_0)$, we get then

$$\left| f'(z) - f_n'(z) \right| = \left| \frac{1}{2i\pi} \int_\gamma \frac{f(w) - f_n(w)}{(w-z)^2} dw \right|$$

$$\leq \frac{1}{2\pi} \times 2\pi\rho \times \sup_{w \in C_\rho(z_0)} \left| \frac{f(w) - f_n(w)}{(w-z)^2} \right|$$

$$\leq \frac{\rho}{(\rho-r)^2} \sup_{w \in C_\rho(z_0)} |f(w) - f_n(w)|$$

(since $|w - z| \geq \rho - r$ if $w \in C_\rho(z_0)$, $z \in \bar{D}_r(z_0)$),

and this gives the result since $f_n(w) \longrightarrow f(w)$ uniformly on the compact set $C_\rho(z_0)$.

$\square$

<u>Example.</u> Let $e : \mathbb{C} \longrightarrow \mathbb{C}$ be defined by $e(z) = e^{2i\pi z}$; in particular note that

$$e(z+w) = e(z)\, e(w), \qquad e(z+1) = e(z), \text{ and}$$

$$|e(z)| = |e^{2i\pi z}| = e^{\mathrm{Re}(2i\pi z)} = e^{-2\pi \mathrm{Im}(z)}$$

Now let $U = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \} \subset \mathbb{C}$.

This is an open set.

We define, for $z \in U$, the <u>theta-function</u>

$$\theta(z) = \sum_{n=0}^{+\infty} e(n^2 z).$$

We claim that $\theta : U \longrightarrow \mathbb{C}$ is well-defined (i.e. the series converges) and holomorphic. To see this, it suffices according to the above to prove that the series converges uniformly in compact subsets of U, since the functions $e(n^2 z)$ are all holomorphic (even on $\mathbb{C}$). But for $z = x+iy$,

$$|e(n^2 z)| = e^{-2\pi n^2 y} \leq e^{-2\pi n y}$$

and $0 \leq e^{-2\pi y} < 1$ since $y > 0$, so that the series converges by comparison with a geometric series, and does so even uniformly on any subset of the form $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq \delta\}$, where $\delta > 0$. Since any closed bounded subset of U is contained in such a set, this implies the result.

We will come back soon to the special function in this example.

6 _ Functions defined by integrals

Theorem _ (II. 5. 4)

Let $U \subset \mathbb{C}$ be open, $I = [a,b] \subset \mathbb{R}^2$ a closed bounded interval. Let

$$F: \quad U \times I \longrightarrow \mathbb{C}$$

be a continuous function such that for every $t_0 \in I$ the function $f_{t_0}(z) = F(z, t_0)$ is in $\mathcal{H}(U)$. Then

$$f(z) = \int_a^b F(z,t)\,dt$$

defines a function $f \in \mathcal{H}(U)$.

Furthermore, we have then

$F'(z,t) = f_t'(z)$

$$f'(z) = \int_a^b F'(z,t)\,dt$$

for all $z \in U$, where $g(t) = F'(z,t)$ is continuous for $z \in U$.

Proof: We use approximations by Riemann sums: for $n \geq 1$, let

$$f_n(z) = \frac{b-a}{n} \sum_{j=0}^{n-1} F\left(z, a + j\,\frac{b-a}{n}\right).$$

Then $f_n \in \mathcal{H}(U)$ (it is a finite linear combination of f_{t_j}, $t_j = a + j\,\frac{b-a}{n}$), so it is enough to prove that f_n converges to f uniformly on compact sets $K \subset U$. To do this, we use the uniform continuity of F restricted to the compact set $K \times I \subset \mathbb{C} \times \mathbb{R}$: for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (z_1, t_1) and (z_2, t_2) in $K \times I$, the conditions

$$|z_1 - z_2| < \delta, \quad |t_1 - t_2| < \delta$$

imply $\quad |F(z_1, t_1) - F(z_2, t_2)| < \dfrac{\varepsilon}{b-a}$.

Now pick n such that $\dfrac{b-a}{n} < \delta$. Then for all $z \in K$, we get

$$f(z) - f_n(z) = \sum_{j=0}^{n-1} \int_{a + j\frac{b-a}{n}}^{a+(j+1)\frac{b-a}{n}} \Big(F(z,t) - \underbrace{F\big(z, a + j\tfrac{b-a}{n}\big)}_{\text{independent of } t,\ \text{so the } \int \text{ is } \frac{b-a}{n} F(z,\, a+j\frac{b-a}{n})} \Big)\, dt$$

and since the z-arguments are equal and $\left| t - \left(a + j\frac{b-a}{n}\right)\right| \leq \dfrac{b-a}{n} < \delta$ for t in the interval $\left[a + j\frac{b-a}{n},\ a + (j+1)\frac{b-a}{n}\right]$, this gives

$$|f(z) - f_n(z)| \leq \frac{\varepsilon}{b-a} \sum_{j=0}^{n-1} \frac{b-a}{n} = \varepsilon.$$

This precisely gives uniform convergence on K.

To compute f', we use the corollary: we have

$$f'(z) = \lim_{n \to \infty} f_n'(z), \qquad z \in U$$

and $$f_n'(z) = \frac{b-a}{n} \sum_{j=0}^{n-1} F'\left(z, a + j\,\frac{b-a}{n}\right)$$

which is the Riemann sum for $\int_a^b F'(z,t)\,dt$. But to check that this exists, we first fix $z \in U$ and check that $g(t) = F'(z,t) = f_t'(z)$ is continuous on I. In fact, let $z \in U$ and $r > 0$ such that $\overline{D_r}(z) \subset U$. Then for t_1, t_2 in $[a,b]$, we have Cauchy's Formula

$$F'(z, t_i) = \frac{1}{2i\pi} \int_\gamma \frac{F(w,t_i)}{(w-z)^2}\,dw$$

[$\gamma = C_r(z)$ oriented counterclockwise] so that

$$F'(z,t_1) - F'(z,t_2) = \frac{1}{2i\pi} \int_\gamma \left(\frac{F(w,t_1)}{(w-z)^2} - \frac{F(w,t_2)}{(w-z)^2} \right) dw$$

$$= \frac{1}{2i\pi} \int_\gamma \frac{F(w,t_1) - F(w,t_2)}{(w-z)^2}\,dw$$

and

$$|F'(z,t_1) - F'(z,t_2)| \leq \frac{1}{2\pi} \sup_{w \in C_r(z)} |F(w,t_1) - F(w,t_2)| \times \frac{2\pi}{r}.$$

So if t_1, t_2 are chosen as above so that $|t_1 - t_2| < \delta$, we get $|F'(z,t_1) - F'(z,t_2)| \leq \frac{\varepsilon}{r}$. (*)

This is enough to prove continuity. Now we get as before

$$\left| \int_a^b F'(z,t)\,dt - f_n'(z) \right| = \left| \sum_{j=0}^{n-1} \int_{a+j\frac{b-a}{n}}^{a+(j+1)\frac{b-a}{n}} \left(F'(z,t) - F'\left(z, a+j\frac{b-a}{n}\right)\right) dt \right|$$

and this converges to 0 (here z is fixed!) using the inequality (*) above.

$\square$

Remark. Under suitable conditions, this can be extended also to "improper" integrals, for instance over $[0, +\infty[$ on $\mathbb{R}$. This is however much easier with the "Lebesgue integral".

Example. Let

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{it\sin(z)}\,dt$$

where $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ for $z \in \mathbb{C}$.

Since $\sin$ is holomorphic on $\mathbb{C}$ and F defined by

$$F(z,t) = e^{it\sin(z)}$$

is continuous on $\mathbb{C} \times \mathbb{R}$, we deduce that J_0

is in $\mathcal{H}(\mathbb{C})$. Furthermore

$$J_0'(z) = \frac{1}{2\pi} \int_0^{2\pi} it\cos(z)\, e^{it\sin(z)}\, dt$$

$$= \frac{i\cos(z)}{2\pi} \int_0^{2\pi} t\, e^{it\sin(z)}\, dt.$$