Chapter V

Meromorphic functions and residue calculus

Cauchy's Theorem and formula compute two types of line integrals:

$$\int_\gamma f(z)\,dz = 0 \quad , \quad f \text{ hol. } \underline{\text{inside}}\ \gamma$$

$$\frac{1}{2i\pi}\int_{\circlearrowleft} \frac{f(w)}{w-z}\,dw = f(z), \quad f \text{ hol. inside disc}$$

We now want to generalize this to functions which are for instance holomorphic except at a finite number of points "inside" some curve.

1 - Singularities ; poles

Definition - (1) Let $z_0 \in \mathbb{C}$ and U an open set containing z_0. A holomorphic function $f \in \mathcal{H}(U - \{z_0\})$ is said to have an isolated singularity at

z_0 (or a "possible" singularity).

(2) If $f \in \mathcal{H}(U - \{z_0\})$ and there exists $r > 0$, such that $D_r(z_0) \subset U$, and $g \in \mathcal{H}(D_r(z_0))$ such that $\left\{ \begin{array}{l} f(z) \neq 0 \text{ and} \\ g(z) = \dfrac{1}{f(z)} \end{array} \right\}$ for all $\left| \begin{array}{l} z \in D_r(z_0), \\ z \neq z_0 \end{array} \right.$

then we say that f has <u>(at most) a pole at z_0.</u>

<u>Example</u> - $z_0 \in \mathbb{C} = U$, $k \geq 0$ integer

$$f(z) = \frac{1}{(z-z_0)^k}, \quad z \neq z_0$$

This has a pole since $\dfrac{1}{f(z)} = (z-z_0)^k$ for $z \neq z_0$, and this holomorphic on $\mathbb{C}$.

On the other hand, $f_1(z) = \exp\left(\frac{1}{z}\right)$ for $z \neq 0$ has an isolated singularity at 0 but this is not a pole. Indeed if $g_1 \in \mathcal{H}(D_r(0))$ existed (with $r > 0$) such that $g_1(z) = \dfrac{1}{f_1(z)}$ for $z \neq 0$, then we would have in particular

$$\lim_{z \to 0} g_1(z) = g_1(0) \qquad [\text{continuity}]$$

but the left-hand side is

$$\lim_{z \to 0} \exp\left(-\frac{1}{z}\right)$$

which does not exist (for instance if $z = \varepsilon > 0$ tending to 0 then $\exp\left(-\frac{1}{z}\right) = \exp\left(-\frac{1}{\varepsilon}\right) \to 0$ but if $z = -\varepsilon < 0$ then $\exp\left(-\frac{1}{z}\right) = \exp\left(\frac{1}{\varepsilon}\right) \to +\infty$).

Proposition. Let $U \subset \mathbb{C}$ open, $z_0 \in \mathbb{C}$, $f \in \mathcal{H}(U - \{z_0\})$. Then f has a pole at z_0 if and only if there exists $r > 0$ such that $D_r(z_0) \subset U$, an integer $k \geq 0$, and $h \in \mathcal{H}(D_r(z_0))$ with $h(z_0) \neq 0$ such that

$$f(z) = \frac{1}{(z - z_0)^k} h(z) \qquad (*)$$

for all $z \in D_r(z_0)$.

The integer k is unique and is called the order of the pole of f at z_0.

Proof. If we have an expression $(*)$ then

f has a pole at z_0: since $h(z_0) \neq 0$, we can find $s \leq r$ such that $h(z) \neq 0$ for $|z - z_0| < s$; then for $\begin{cases} z \neq z_0 \\ z \in D_s(z_0) \end{cases}$, we get

$$f(z) \neq 0 \quad \text{and} \quad \frac{1}{f(z)} = (z - z_0)^k \frac{1}{h(z)},$$

and the function $g(z) = (z - z_0)^k h(z)^{-1}$ is in $\mathcal{H}(D_s(z_0))$.

Conversely, suppose f has a pole at z_0, with $g \in \mathcal{H}(D_r(z_0))$ such that $g(z) = \dfrac{1}{f(z)}$ if $z \in D_r(z_0)$ (in particular $f(z) \neq 0$ in this disc).

Let $k = \mathrm{ord}_{z_0}(g)$; there exists $\begin{cases} s > 0 \\ s \leq r \end{cases}$ and $\tilde{g} \in \mathcal{H}(D_s(z_0))$ with $\tilde{g}(z_0) \neq 0$ such that

$$\forall z \in D_s(z_0), \quad g(z) = (z - z_0)^k \tilde{g}(z).$$

Pick further $s' \leq s$, $s' > 0$, such that $\tilde{g}(z)$ is non-zero for $z \in D_{s'}(z_0)$; then we get $h = \dfrac{1}{\tilde{g}} \in \mathcal{H}(D_{s'}(z_0))$, $h(z_0) \neq 0$, and

$$\frac{1}{f(z)} = (z - z_0)^{-k} h(z)$$

for $z \neq z_0$ in $D_{r'}(z_0)$, which is of the type $(*)$.

$\square$

Notation - It will be convenient to denote $D_r^*(z_0) = \{ z \in D_r(z_0) \mid z \neq z_0 \}$; this is called a "punctured (open) disc" centered at z_0.

2 - Removable singularities

The following simple but powerful result allows us to detect when a singularity is not really one.

(III.3.1)

Theorem - Let $U \subset \mathbb{C}$ be open, $z_0 \in U$. Let $f \in \mathcal{H}(U - \{z_0\})$.

There exists $g \in \mathcal{H}(U)$ which coincides with f outside z_0 [then we say that z_0 is a removable singularity] if and only if there exists $r > 0$

such that f is bounded on $D_r^\circ(z_0)$.

Proof. If g exists, then it is continuous, so it is bounded on $\overline{D_r}(z_0)$ if $\overline{D_r}(z_0) \subset U$, and then so is f on $D_r^\circ(z_0) \subset \overline{D_r}(z_0)$.

Conversely, suppose f is bounded on some $D_r^\circ(z_0)$ with $r > 0$.

Define $\tilde{f} : D_r(z_0) \longrightarrow \mathbb{C}$ by

$$\begin{cases} \tilde{f}(z) = (z - z_0)^2 f(z), & z \neq z_0 \\ \tilde{f}(z_0) = 0 \end{cases}$$

Then $\tilde{f} \in \mathcal{H}(D_r(z_0))$: it is clear that $\tilde{f}'(z)$ exists if $z \neq z_0$, and for z_0, we have for $h \neq 0$, $|h| < r$

$$\frac{\tilde{f}(z_0 + h) - \tilde{f}(z_0)}{h} = \frac{h^2 f(z_0 + h)}{h}$$
$$= h f(z_0 + h),$$

which converges to 0 as $h \to 0$ since f is bounded on $D_r(z_0)$. So indeed $\tilde{f}$ is ho-

-lomorphic on $D_r(z_0)$, and in fact $\tilde{f}'(z_0) = 0$.

Now $\tilde{f}(z_0) = \tilde{f}'(z_0) = 0$ mean that the order of vanishing k of $\tilde{f}$ at z_0 is ≥ 2.

If $k = +\infty$, then $\tilde{f}(z) = 0$ for $z \in D_r(z_0)$, so $f(z) = 0$ on $D_r^*(z_0)$, and we can take $g(z) = 0$.

Otherwise, let $\tilde{f}(z) = (z - z_0)^k \tilde{f}_1(z)$ with $\tilde{f}_1$ holomorphic close to z_0; we get

$$f(z) = \frac{\tilde{f}(z)}{(z - z_0)^2} = (z - z_0)^{\overbrace{k-2}^{\geq 0}} \tilde{f}_1(z)$$

for $z \in D_s^*(z_0)$ for some $s > 0$, so we can take $g(z) = (z - z_0)^{k-2} \tilde{f}_1(z) \in \mathcal{H}(D_s(z_0))$ to prolong f. $\square$

Example - Let $f(z) = \dfrac{z}{e^z - 1}$, $z \neq 0$, $z \in D_1(0)$.

Then f has a removable singularity at 0 since $\lim\limits_{z \to 0} \dfrac{z}{e^z - 1} = \dfrac{1}{\exp'(0)} = 1$ implies that f is bounded in a neighborhood of 0.

Note - If z_0 is a removable singularity of f, then $\lim\limits_{\substack{z \to z_0 \\ z \neq z_0}} f(z)$ exists, and the unique g holomorphic at z_0 that extends f is determined by $g(z_0) = \lim\limits_{\substack{z \to z_0 \\ z \neq z_0}} f(z)$.

3 _ Principal part and residue

Proposition _ [III . 1.3] Let $U \subset \mathbb{C}$ open, $z_0 \in \mathbb{C}$ and $f \in \mathcal{H}(U)$ with a pole of order $k \geq 0$ at z_0. There exist unique complex numbers

$$a_k \neq 0, \quad a_{k-1}, \quad \ldots, \quad a_1$$

and $g \in \mathcal{H}(D_r(z_0))$ for some $r > 0$ such that

$$f(z) = g(z) + \sum_{j=1}^{k} \frac{a_j}{(z-z_0)^j}$$

for all $z \in \mathring{D}_r(z_0)$.

Definition _ The number a_1 above is called the residue of f at z_0, denoted $\operatorname{res}_{z_0}(f)$.
The function $\sum\limits_{j=1}^{k} \frac{a_j}{(z-z_0)^j}$ is called the principal part at z_0.

Proof. According to Section 1, we can write $f(z) = (z - z_0)^{-k} h(z)$ in some $D_r^*(z_0)$ with $r > 0$, $h \in \mathcal{H}(D_r(z_0))$ such that $h(z_0) \neq 0$. We expand h in power series: since $D_r(z_0) \subset U$, we have

$$h(z) = \sum_{n=0}^{+\infty} \frac{h^{(n)}(z_0)}{n!} (z - z_0)^n$$

in $D_r(z_0)$, and then for $z \in D_r^\circ(z_0)$:

$$f(z) = \frac{1}{(z - z_0)^k} \left(h(z_0) + h'(z_0)(z - z_0) + \dots \right)$$

$$= \underbrace{\sum_{j=0}^{k} \frac{1}{(k-j)!} \frac{h^{(k-j)}(z_0)}{(z - z_0)^j}}_{\text{principal part}} + \underbrace{\sum_{n=k+1}^{+\infty} (z - z_0)^{n-k} \frac{h^{(n)}(z_0)}{n!}}_{g(z)}.$$

$\square$

Example. If the order of the pole is 1, then

$$\operatorname{res}_{z_0}(f) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} (z - z_0) f(z).$$

(Indeed,

$$f(z) = \frac{a_1}{z - z_0} + g(z)$$

for $z \in D_r^\circ(z_0)$ implies that

$$(z-z_0)f(z) = a_1 + (z-z_0)g(z) \xrightarrow[z \to z_0]{} a_1 \Big)$$

In fact, if $\lim\limits_{\substack{z \to z_0 \\ z \neq z_0}} (z-z_0)f(z)$ exists <u>and is $\neq 0$</u>, then f has a pole at z_0 of order 1, with that limit as residue. If the limit exists but is 0, then f has a removable singularity at z_0. (Indeed, the condition implies that $\tilde{f}(z) = (z-z_0)f(z)$ has a removable singularity, so "is" holomorphic on some $D_1(z_0)$; if $\tilde{f}(z_0) \neq 0$, then

$$f(z) = \frac{1}{z-z_0}\tilde{f}(z)$$

shows that f has a pole of order 1, and if $\tilde{f}(z_0) = 0$, then $\mathrm{ord}_{z_0}(\tilde{f}) \geq 1$, so that

$$\tilde{f}(z) = (z-z_0)\tilde{f}_1(z), \quad z \in D_s(z_0)$$

$$\tilde{f}_1 \in \mathcal{H}(D_s(z_0))$$

and $\tilde{f}_1$ extends f.)

4. The residue formula

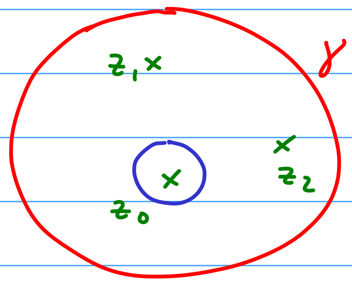
Theorem. Let $U \subset \mathbb{C}$ open, $F \subset U$ a finite set and $f \in \mathcal{H}(U - F)$ holomorphic except at points of F. Let γ be a circle contained in U, with counterclockwise orientation such that $\gamma \cap F = \emptyset$.
Let $D \subset U$ be the open interior of the disc bounded by γ.
Assume that each $z_0 \in F$ is a pole of f.
Then
$$\int_\gamma f(z)\,dz = 2i\pi \sum_{z_0 \in F \cap D} \operatorname{res}_{z_0}(f).$$

Proof. For each $z_0 \in F$, let p_{z_0} be the principal part at z_0. Note that p_{z_0} is holo-morphic on $\mathbb{C}$ except at z_0. Now define
$$g(z) = f(z) - \sum_{z_0 \in F} p_{z_0}(z)$$

if $z \notin F$. Then $g \in \mathcal{H}(U - F)$.

We claim that in fact g can be extended holo-morphically to U.

Indeed, let $z_0 \in F$ and let $r > 0$ be such that $D_r(z_0) \subset D$ and $D_r(z_0) \cap F = \emptyset$.



Then for $z \in D_r^\circ(z_0)$,

$$g(z) = \sum_{\substack{z_1 \in F \\ z_1 \neq z_0}} p_{z_1}(z) + \underbrace{f(z) - p_{z_0}(z)}_{\text{extends to } D_r(z_0)}$$

in $\mathcal{H}(D_r(z_0))$

so g extends to a holomorphic function on $D_r(z_0)$.

Since $g \in \mathcal{H}(U)$, Cauchy's Theorem gives

$$\int_\gamma g(z)\, dz = 0$$

so that

$$\int_\gamma f(z)\, dz = \sum_{z_0 \in F} \int_\gamma p_{z_0}(z)\, dz.$$

This means we have reduced the problem to

The case of the principal parts.

Pick $z_0 \in F \cap D$ and write

$$p_{z_0}(z) = \frac{a_k}{(z-z_0)^k} + \dots + \frac{a_1}{z-z_0}.$$

Then

$$\int_\gamma p_{z_0}(z)\,dz = \sum_{j=1}^{k} \int_\gamma \frac{a_j}{(z-z_0)^j}\,dz$$

$$= \sum_{j=1}^{k} \frac{2i\pi}{j!}\, a_j\, \varphi^{(j-1)}(z_0)$$

by Cauchy's Formula, where $\varphi(z) = 1$, so

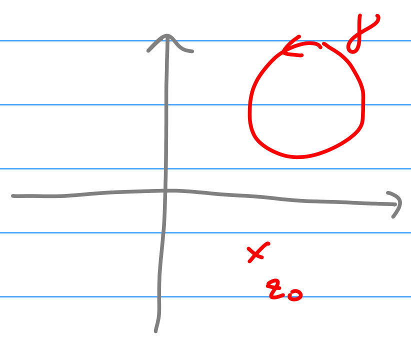
$$= 2i\pi\, a_1 = 2i\pi \operatorname{res}_{z_0}(f).$$

If $z_0 \in F$ but $z_0 \notin D$, then on the other hand

$$\int_\gamma p_{z_0}(z)\,dz = \sum_{j=1}^{k} \int_\gamma \frac{a_j}{(z-z_0)^j}\,dz$$

$$= 0$$

because in that case the function $\dfrac{1}{(z-z_0)^j}$ is holomorphic inside the disc D.



Remark.

(1) It would be enough to assume that the points $z_0 \in F \cap D$ are poles, not necessarily those outside D. To see this, argue as above but with

$$g(z) = f(z) - \sum_{z_0 \in F \cap D} p_{z_0}(z).$$

Then $g \in \mathcal{H}(U - F_1)$ where $F_1 = F - (F \cap D)$ by the same argument. Since $U - F_1$ is an open set containing D, we get again

$\int_\gamma g(z)\,dz = 0$ by Cauchy's Theorem.

(2) If γ is not a circle but, say, a triangle or a rectangle, to see that the formula still holds, one must only check that Cauchy's Theorem is true along γ ($\int_\gamma g(z)\,dz = 0$ if g holomorphic) and check that

$$\int_\gamma \frac{1}{(z-z_0)^j}\,dz = \begin{cases} 2i\pi, & j = 1 \\ 0, & j \neq 1 \end{cases}$$

holds if $z_0 \in D$. For $j \neq 1$, this is true for any closed curve simply because $\frac{1}{(z-z_0)^j}$ has the primitive

$\frac{1}{1-j} \frac{1}{(z-z_0)^{j-1}}$ on $\mathbb{C} - \{z_0\}$.

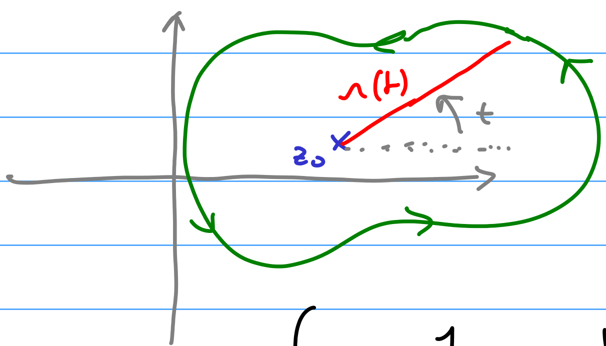
For $j = 1$, we can now at least prove it

wherever γ has a parameterization

$$\gamma \left\{ \begin{array}{ccc} [a, b] & \longrightarrow & \mathbb{C} - \{z_0\} \\ t & \longmapsto & z_0 + r(t) e^{i\theta(t)} \end{array} \right.$$

for some C^1 functions r on $[0, 2\pi]$ such that $r(a) = r(b)$, $r(t) > 0$ for all t, and θ is C^1 with $\theta(a) = 0$, $\theta(b) = 2\pi$.

(Such curves are "going once around z_0 in the counterclockwise direction", and are quite general; for instance this works for rectangles, triangles, ...)



Indeed, we then compute

$$\gamma'(t) = r'(t) e^{i\theta(t)} + i\theta'(t) r(t) e^{i\theta(t)}$$

so

$$\int_\gamma \frac{1}{(z-z_0)} dz = \int_a^b \frac{1}{r(t) e^{it}} \gamma'(t) \, dt$$

$$= \int_a^b \frac{1}{r(t)\cancel{e^{i\theta(t)}}} r'(t)\, \cancel{e^{i\theta(t)}}\, dt$$

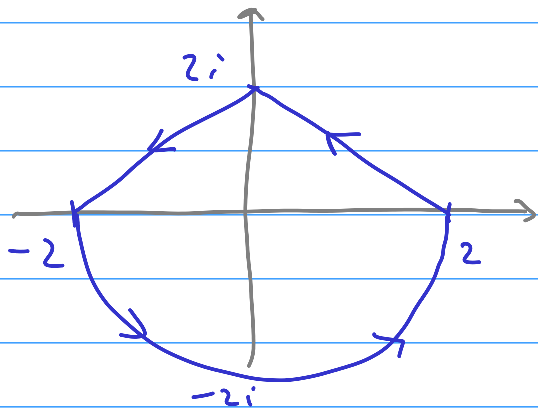
$$+\, i \int_a^b \frac{1}{\cancel{r(t)}\,\cancel{e^{i\theta(t)}}} \cancel{r(t)}\, \theta'(t)\, \cancel{e^{i\theta(t)}}\, dt$$

$$= \Big[\log r(t) \Big]_a^b + i \Big[\theta(t) \Big]_a^b = i(2\pi - 0).$$

<u>Examples.</u> Before going further in the theory, we consider some simple examples.

(1) Consider the line integral

$$\int_\gamma \frac{\exp(z^2)}{(z-1)(z+1)}\, dz$$ where γ is the following curve:



The function

$$f(z) = \frac{\exp(z^2)}{(z-1)(z+1)^2}$$ is

holomorphic on $\mathbb{C} - \{-1, 1\}$, and has

(i) a pole of order 1 at $z = 1$ (since

$$\lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{\exp(z^2)}{(z+1)^2} = \frac{e}{4}$$ is

finite), with residue $\frac{e}{4}$.

(2) a pole of order 2 at $z = -1$ (since $\frac{1}{f(z)} = (z+1)^2 g(z)$, $g(z) = \frac{z-1}{\exp(z^2)}$ holo-morphic around -1 with $g(-1) = \frac{-2}{e} \neq 0$)

Hence

$$\int_\gamma f(z)\,dz = 2i\pi \left(\operatorname{res}_1 f + \operatorname{res}_{-1} f \right).$$

To compute $\operatorname{res}_{-1} f$ we use the expansion of g around -1 to compute the principal part:

$$f(z) = \frac{1}{(z+1)^2} \frac{\exp(z^2)}{(z-1)} = \frac{1}{(z+1)^2} \left(-\frac{e}{2} + \alpha(z+1) + \dots \right)$$

where $\alpha = \left(\frac{\exp(z^2)}{z-1} \right)'(-1)$

$$= \left(\frac{2ze^{z^2}(z-1) - e^{z^2}}{(z-1)^2} \right)(-1) = \frac{3e}{4},$$

so the principal part is

$$-\frac{e}{2} \frac{1}{(z+1)^2} + \frac{3e}{4} \frac{1}{(z+1)}$$

and hence $\operatorname{res}_{-1}(f) = \frac{3e}{4}$. So the integral is

$$\int_\gamma f(z)\,dz = 2i\pi \left(\frac{e}{4} + \frac{3e}{4} \right) = 2ie\pi.$$

(2) Now we use residues to compute a <u>real</u> integral: consider

$$I_n = \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^n}, \quad n \geq 1$$

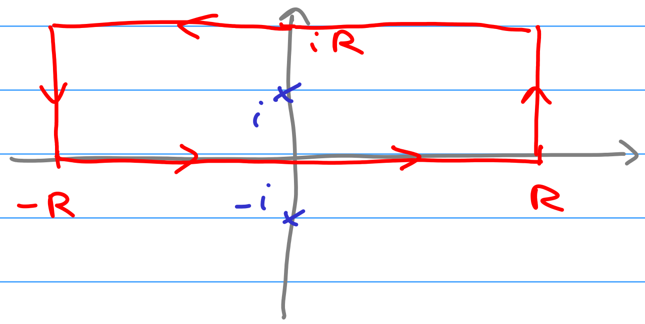
$$= \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{(1+x^2)^n}$$

(the integral exists by comparison with $\frac{1}{x^{2n}}$ at $\pm\infty$)

We note that $\frac{1}{(1+x^2)^n} = f(x)$ where $f(z) = \frac{1}{(1+z^2)^n}$ is holomorphic except at $z = i$ and $z = -i$, where it has poles of order n (e.g. $\frac{1}{f(z)} = (z-i)^n \, (z+i)^n$ with $(i+i)^{2n} \neq 0$).

The idea is to write an easily evaluated line integral, one part of which is $\int_{-R}^{R} f(x)dx$.

Consider γ_R as follows:



Then $\displaystyle\int_\gamma f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\uparrow} f + \int_{\leftarrow} f + \int_{\downarrow} f.$

Each of the remaining three segments have length $\leq 2R$ and $|(1+z^2)^n| \geq (R-1)^n R^n$

so $\displaystyle\left|\int_{\uparrow} f + \int_{\leftarrow} f + \int_{\downarrow} f\right| \leq \frac{4R}{R^{n-1}(R-1)^n} \xrightarrow[R\to+\infty]{} 0$ (because $n \geq 1$)

(length $= 4R$)

Hence

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz$$

$$= 2i\pi \operatorname{res}_{z=i} f(z).$$

To compute the residue: write

$$f(z) = \frac{1}{(z-i)^n}\, g(z), \quad g(z) = \frac{1}{(z+i)^n}, \text{ and}$$

expand g around i. For this it is useful to observe that $g(z) = \dfrac{(-1)^{n-1}}{(n-1)!} h^{(n-1)}(z)$

where $h(z) = \dfrac{1}{z+i}$. Since

$$h(z) = \frac{1}{2i\left(1 - \frac{1-z}{2i}\right)} = \frac{1}{2i} \sum_{k=0}^{+\infty} \frac{(1-z)^k}{(2i)^k},$$

we get

$$h'(z) = \frac{1}{2i} \sum_{k=1}^{+\infty} (-1)^k (z-i)^{k-1} \frac{k}{(2i)^k}$$

$$\vdots$$

$$g(z) = \frac{(-1)^{n-1}}{(n-1)!} h^{(n-1)}(z) = \frac{(-1)^{n-1}}{2(n-1)!\, i} \sum_{k=n-1}^{+\infty} (-1)^k (z-i)^{k-(n-1)}$$

$$\times \frac{k\ldots(k-n+1)}{(2i)^k}.$$

To compute $res_i(f)$ we need the coefficient of $(z-i)^{n-1}$, so we take $k = 2n-2$

and

$$res_i(f) = \frac{(-1)^{n-1}}{2i(n-1)!} \times \overbrace{\cancel{(-1)^{2n-2}}}^{=1} \times \frac{(2n-2)\ldots(n-1)}{\underbrace{\cancel{(2i)^{2n-2}}}_{=2^{2n-2}(-1)^{n-1}}}$$

$$= \frac{1}{2^{2n-2}.(2i)} \frac{(2n-2)!}{((n-1)!)^2}$$

hence

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2i\pi\, res_i f = \frac{(2n-2)!\,\pi}{2^{2n-2}((n-1)!)^2}$$

(as announced in Chapter 1).

(3) We can also "recover" the Cauchy formula:

consider $f \in \mathcal{H}(U)$, $z_0 \in U$, $\overline{D}_r(z_0) \subset U$, $z \in D_r(z_0)$ and

$$I = \int_{C_r(z_0)} \frac{f(w)}{w-z}\,dw.$$

counterclockwise $\longrightarrow$ $C_r(z_0)$

So we are taking $F = \{z\}$ and

$$g(w) = \frac{f(w)}{w - z}, \quad g \in \mathcal{H}(U - \{z\}).$$

The function g has a pole of order 1 with residue $f(z)$ at $w = z$ if $f(w) \neq 0$: then

$$(w - z)\, g(w) = f(w) \xrightarrow[w \neq z]{w \to z} f(z) \neq 0.$$

So in that case

$$I = 2i\pi\, f(z).$$

But note that this is true also if $f(w) = 0$ (both sides are then zero). It is in fact conve-nient to define $\operatorname{res}_{z_0}(f) = 0$ if f has a removable singularity at z_0.

5 - Meromorphic functions

Lemma. Let $U \subset \mathbb{C}$ be open, $z_0 \in U$ and $f \in \mathcal{H}(U - \{z_0\})$. Then f has a pole of order $k \geq 1$ at z_0 $\iff$ $\lim\limits_{\substack{z \to z_0 \\ z \neq z_0}} |f(z)| = +\infty$.

Proof: If f has a pole of order $k \geq 1$, then $f(z) = \frac{h(z)}{(z-z_0)^k}$ on $D_r^{\circ}(z_0)$ for some $r > 0$ and $h \in \mathcal{H}(D_r(z_0))$ such that $h(z_0) \neq 0$.

Then $|f(z)| = |h(z)| \, |z - z_0|^{-k}$ and this tends to $+\infty$ because $|h(z)| \to |h(z_0)| \neq 0$ and $k \geq 1$.

Conversely, if $|f(z)| \longrightarrow +\infty$ then we can find $r > 0$ s.t. $|f(z)| \geq 1$ for $z \in D_r^{\circ}(z_0)$.

We define then $g(z) = \frac{1}{f(z)}$, $z \in D_r^{\circ}(z_0)$

Because $|g(z)| = \frac{1}{|f(z)|} \leq 1$ in $D_r^{\circ}(z_0)$, the function has a removable singularity at z_0; it extends to a holomorphic function on $D_r(z_0)$ by defining $g(z_0) = \lim \frac{1}{f(z)} = 0$.

So, by definition, f has a pole at z_0.

It must have order $k \geq 1$ because otherwise f would be bounded as $z \longrightarrow z_0$. $\square$

This proposition suggests the following defi-
-nitions:

unsigned infinity

Definition - (1) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

(2) A sequence (z_n) in $\hat{\mathbb{C}}$ *converges to ∞* if

in $\mathbb{R}$

$\lim |z_n| = +\infty$, and a function $f(z) \longrightarrow \infty$

si $\lim |f(z)| = +\infty$.

(3) Let $U \subset \mathbb{C}$ open. A *meromorphic function* f on U is a function $f : U \longrightarrow \hat{\mathbb{C}}$ such that:

(i) The set $S_f = \{ z \in U \mid f(z) = \infty \}$ has no accumulation point <u>in U</u> [$\Leftrightarrow$ if $K \subset U$ is compact, then $S_f \cap K$ is finite]

(ii) let $U_f = \{ z \in U \mid f(z) \neq \infty \}$;
then f restricted to U_f is holomorphic.

(iii) for any z_0 s.t. $f(z_0) = \infty$, the function f has a pole of order $\geqslant 1$ at z_0.

We denote by $\mathcal{M}(U)$ the set of functions that are meromorphic on U.

Example. Let $p_1 \in \mathbb{C}[X]$ be a polynomial and $p_2 \neq 0$ in $\mathbb{C}[X]$ a non-zero polynomial. Assume p_1, p_2 do not have a common zero. Then define

$$f(z) = \begin{cases} \dfrac{p_1(z)}{p_2(z)}, & p_2(z) \neq 0 \\ \infty, & p_2(z) = 0 \end{cases}$$

Then $f \in \mathcal{M}(\mathbb{C})$. Indeed, f is holomorphic for z outside of the finite set of zeros of p_2, and if $p_2(z_0) = 0$, then

$$|f(z)| = \frac{|p_1(z)|}{|p_2(z)|} \longrightarrow +\infty$$

$p_2(z_0) \neq 0$

0

so (by the first lemma), the function f has a pole at z_0.

Proposition. $U \subset \mathbb{C}$ open

(1) $\mathcal{M}(U) \supset \mathcal{H}(U)$

(2) $\mathcal{M}(U)$ is a vector space

(3) Elements of $\mathcal{M}(U)$ can be multiplied

(4) If $f \in \mathcal{M}(U)$ is not zero, then $\frac{1}{f}$ is in $\mathcal{M}(U)$.

(Explanation: to add or multiply two mero-morphic functions f, g, add or multiply them at the points which are not poles; then extend them to poles by taking the limit in $\hat{\mathbb{C}}$.)

Proof. Consider (2): $\alpha f + \beta g \in \mathcal{H}(U_f \cap U_g)$ so we only need to consider z_0 which is a pole of f, or g, or both. Write

$$f(z) = p(z) + \tilde{f}(z)$$

$$g(z) = q(z) + \tilde{g}(z)$$

with $\tilde{f}, \tilde{g}$ holomorphic on $D_r(z_0)$, p, q the principal parts. So

$$\alpha f(z) + \beta g(z) = \underbrace{\left(\alpha p(z) + \beta q(z)\right)}_{\text{combination of } \frac{1}{(z-z_0)^j}} + \underbrace{\left(\alpha \tilde{f}(z) + \beta \tilde{g}(z)\right)}_{\in \mathcal{H}(D_r(z_0))}$$

and so $\alpha f + \beta g$ has a pole at z_0, of order ≥ 1 unless $\alpha p + \beta q = 0$ (which may happen).

Similarly for $f \cdot g$.

Now for $1/f$: if $z_0 \in U_f$ and $f(z_0) \neq 0$, then $1/f$ is holomorphic at z_0; if $z_0 \in U_f$ and $f(z_0) = 0$, then $1/f$ has a pole of order $k = \mathrm{ord}_{z_0}(f) \geq 1$; if z_0 is a pole of f, then $1/f$ has a removable singularity (since $\dfrac{1}{|f(z)|} \xrightarrow[z \to z_0]{} 0$ then).

$\square$

<u>Example</u>. $p_1 \in \mathbb{C}[X]$, $p_2 \in \mathbb{C}[X]$ non zero

$p_1 \cdot \dfrac{1}{p_2} = \dfrac{p_1}{p_2} \in \mathcal{M}(\mathbb{C})$.

<u>Definition</u>. $U \subset \mathbb{C}$ open, $z_0 \in U$

If $f \in \mathcal{M}(U)$, $f \neq 0$, define the <u>valuation</u> (or <u>order</u>) of f at z_0, denoted $\mathrm{ord}_{z_0}(f)$ to be the integer $k \in \mathbb{Z}$ such that :

(i) if $f(z_0) \neq \infty$ [z_0 is not a pole] then $k \geq 0$ is the order of vanishing of f at z_0

(ii) if $f(z_0) = \infty$ then $k \leq -1$ is minus the order of the pole at z_0.

Combining what we know about zeros and poles we get :

Proposition - $f \in \mathcal{M}(U)$, $f \neq 0$; $z_0 \in U$

(1) $\mathrm{ord}_{z_0}(f) = k \iff$ there exists $r > 0$ and $h \in \mathcal{H}(D_r(z_0))$ such that $h(z_0) \neq 0$ and

$$f(z) = (z - z_0)^{k} h(z)$$

is < 0 if z_0 is a pole

for $z \in D_r^\circ(z_0)$.

(2) $\mathrm{ord}_{z_0}(fg) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$

(3) If $f + g \neq 0$ then

$$\mathrm{ord}_{z_0}(f+g) \geq \min\left(\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\right)$$

Example. Since $f(z) = e^z - 1$ defines an element of $\mathcal{H}(\mathbb{C})$, $\frac{1}{f(z)} = \frac{1}{e^z-1}$ is in $\mathcal{M}(\mathbb{C})$; it has poles at $z = 2i\pi n$ for all $n \in \mathbb{Z}$, and $\operatorname{ord}_{2i\pi n} \frac{1}{e^z-1} = -1$.

(because $2i\pi n$ is a zero of order 1 of $e^z - 1$: the value of the derivative is $e^{2i\pi n} = 1$)

Consider then $z_0 \in \mathbb{C}$ and

$$\operatorname{ord}_{z_0}\left(\frac{z}{e^z-1}\right) = \operatorname{ord}_{z_0}(z) - \operatorname{ord}_{z_0}(e^z-1)$$

$$= \begin{cases} 0 & \text{if } z_0 \neq 2i\pi n, \\ 0 & \text{if } z_0 = 0 \\ -1 & \text{if } z_n = 2i\pi n, \\ & \quad n \neq 0 \text{ integer} \end{cases}$$

But for instance $\frac{z}{(e^z-1)^2}$ has a pole of order 1 at 0 (and of order 2 at $2i\pi, \ldots$)

6. Values of holomorphic functions

Lemma. Let $U \subset \mathbb{C}$ be $\begin{cases}\text{connected} \\ \text{open},\end{cases}$ $f \in \mathcal{M}(U)$ non-zero. Then $f' \in \mathcal{M}(U)$ and the logarithmic derivative $f'/f \in \mathcal{M}(U)$ has poles of <u>order 1</u> exactly at those $z_0 \in U$ such that $\mathrm{ord}_{z_0}(f) \neq 0$ [so either z_0 is a pole of $f(z_0) = 0$] and it has residue at z_0 equal to $\mathrm{ord}_{z_0}(f)$.

<u>Proof</u> – Clearly $f' \in \mathcal{H}(U_f)$, where U_f is the complement of the set of poles of f.

If z_0 is a pole of f, then from

$$f(z) = \frac{h(z)}{(z-z_0)^j}, \qquad \begin{cases} h \in \mathcal{H}(D_r(z_0)) \\ z \in D_r^\circ(z_0), \ j \geq 1 \\ h(z_0) \neq 0 \end{cases}$$

we get

$$f'(z) = \frac{h'(z)}{(z-z_0)^j} - \frac{j\,h(z)}{(z-z_0)^{j+1}}$$

$$= \frac{\overbrace{(z-z_0)h'(z) - j\,h(z)}^{\text{takes value } -j\,h(z_0) \neq 0}}{(z-z_0)^{j+1}}$$

so f' has a pole at z_0 of order $j+1$. Hence $f' \in \mathcal{M}(U)$ and so also $f'/f \in \mathcal{M}(U)$.

For any $z_0 \in U$, we get

$$\operatorname{ord}_{z_0}(f'/f) = \operatorname{ord}_{z_0}(f') - \operatorname{ord}_{z_0}(f)$$

$$= \begin{cases} -(j+1)-(-j) = -1, & z_0 \text{ pole of order } j \\ j-1-j = -1, & z_0 \text{ zero of order } j \\ 0 & \text{otherwise} \end{cases}$$

so we get poles of order <u>1</u> at each zero or pole of f. To compute the residue, we write again

$$f(z) = (z-z_0)^k \tilde{f}(z), \qquad \begin{array}{l} z \in D_r^\circ(z_0) \\ r > 0 \\ \tilde{f}(z_0) \neq 0 \\ k = \operatorname{ord}_{z_0}(f) \end{array}$$

and then get

$$\frac{f'(z)}{f(z)} = \underbrace{\frac{k}{z-z_0}}_{\text{principal part}} + \frac{\tilde{f}'(z)}{\tilde{f}(z)}$$

holomorphic on $D_r(z_0)$

[because of the formula $\dfrac{(fg)'}{fg} = \dfrac{f'}{f} + \dfrac{g'}{g}$]

so that $\operatorname{res}_{z_0}(f'/f) = \operatorname{ord}_{z_0}(f)$.

$\square$

Proposition - [III. 4. 1] $U \subset \mathbb{C}$

$\gamma \subset U$ circle (or any curve s.t. the residue formula holds), $f \in \mathcal{M}(U)$.

If γ contains no zero or pole of f then

$$\frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{\substack{z_0 \text{ zero} \\ \text{of } f \text{ inside } \gamma}} \operatorname{ord}_{z_0}(f) + \sum_{\substack{z_0 \text{ pole} \\ \text{of } f \text{ inside } \gamma}} \operatorname{ord}_{z_0}(f)$$

$$= \text{nb. of zeros of } f \text{ inside } \gamma \text{ (with multiplicity)} - \text{nb. of poles inside } \gamma \text{ (with multiplicity)}$$

Proof - This follows immediately from the residue theorem and the previous lemma.

$\square$

Example - If $f \in \mathbb{C}[x]$ is non-zero, and $R > 0$ is large enough, then

$$\frac{1}{2i\pi}\int_{C_R(0)} \frac{f'(z)}{f(z)}\,dz = \deg(f).$$

Corollary 1. [Rouché's Theorem, III.4.3]

$U \subset \mathbb{C}$ open, f, g in $\mathcal{H}(U)$

If $\overline{D}_r(z_0) \subset U$ and

$$|f(z)| > |g(z)|, \quad z \in C_r(z_0)$$

then f and $f+g$ have the same number of zeros in $D_r(z_0)$, counted with multiplicity.

Proof. Since there are no poles, we need to prove that if γ is the circle $C_r(z_0)$ counterclockwise, then

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz.$$

Note that the assumption implies that $|f(z)| > 0$ and $|f(z)+g(z)| \geq |f(z)| - |g(z)| > 0$ for $z \in \gamma$, so these integrals are well-defined.

Now we define $\varphi(z) = \dfrac{g(z)}{f(z)}$ for $z \in \gamma$; this is a continuous function with $|\varphi(z)| < 1$ for all $z \in \gamma$.

We have $\dfrac{(f'+g')(z)}{(f+g)(z)} = \dfrac{f'(z)}{f(z)} + \dfrac{\varphi'(z)}{1+\varphi(z)}$ (the right-hand side is $\dfrac{f'+f'\varphi+f\varphi'}{f(1+\varphi)} = \dfrac{f'+f'g/f+f\left(\frac{g'f-gf'}{f^2}\right)}{f+g}$

$$= \frac{f' + \cancel{f'g/f} + g' - \cancel{gf'/f}}{f+g} \Big]$$

and therefore

$$\int_\gamma \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz = \int_\gamma \frac{f'(z)}{g'(z)}\,dz + I$$

where

$$I = \int_\gamma \frac{\varphi'(z)}{1+\varphi(z)}\,dz$$

$$= \int_\gamma \varphi'(z) \sum_{n\geq 0} (-1)^n \varphi(z)^n \, dz$$

[because $|\varphi(z)| < 1$]

$$= \sum_{n\geq 0} (-1)^n \int_\gamma \varphi'(z)\varphi(z)^n\,dz$$

(because φ is continuous on the compact set γ, so there exists $\delta > 0$ s.t. $|\varphi(z)| \leq 1-\delta$ on γ, giving uniform convergence of the series)

$$= 0$$

since $\varphi'\varphi^n$ is the derivative of $\dfrac{1}{n+1}\varphi^{n+1}$,

which is holomorphic in some open set containing

f [namely the open set where $f(z) \neq 0$].

$\square$

This to two important theorems:

Theorem 1. [Open image theorem, III.4.4]

If $f \in \mathcal{H}(U)$ is not constant (and U connected) then for any open set $V \subset U$, the image $f(V) \subset \mathbb{C}$ is open.

(So, for instance, if $f \in \mathcal{H}(D_1(0))$ is not constant, then it is not possible that $f(z) \in \mathbb{R}$ for all z)

Proof. Let $z_0 \in U$ and $w_0 = f(z_0)$. We need to show (assuming f not constant) that any w_1 sufficiently close to w_0 is a value of f.

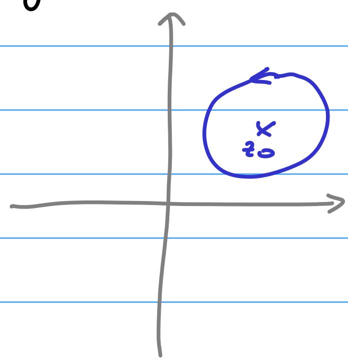
We do this by interpreting a solution of $f(z_1) = w_1$ as a solution of $f(z_1) - w_1 = 0$, and the fact that $f(z_0) = w_0$ as $\tilde{f}(z_0) = 0$, where we put

$\tilde{f}(z) = f(z) - w_0$. So we want to compare the number of zeros of two functions. Note that we have

$$f(z) - w_1 = \underbrace{(f(z) - w_0)}_{\tilde{f}} + \underbrace{(w_0 - w_1)}_{\tilde{g}}.$$

If we manage to ensure that $|\tilde{f}(z)| > |\tilde{g}(z)|$ for z in some circle around z_0, then $\tilde{f} + \tilde{g}$ has a zero w_1 inside the corresponding disc.

Let $r > 0$ be such that $\overline{D}_r(z_0) \subset U$ and such that $f(z) \neq w_0$ on $C_r(z_0)$



[This exists because f is not constant equal to w_0 so z_0 is an isolated zero of $f(z) - w_0$].

Since $C_r(z_0)$ is compact we can find $\delta > 0$ such that $|f(z) - w_0| = |\tilde{f}(z)| \geq \delta$ for $z \in C_r(z_0)$. Now for any $w_1 \in \mathbb{C}$ such that $|w_1 - w_0| < \delta$, we get $|\tilde{f}(z)| > |\tilde{g}(z)|$

on $C_r(z_0)$, so $\tilde{f}$ and $\tilde{f}+\tilde{g}$ have the same number of zeros inside $D_r(z_0)$.

$\square$

<u>Theorem 2</u> - [Maximum modulus principle, III. 4.5, 4.6]

Let $U \subset \mathbb{C}$ be open and connected. Let $f \in \mathcal{H}(U)$ be non constant. Then there is no $z_0 \in U$ s.t. $|f(z)| \leq |f(z_0)|$ for $z_0 \in U$.

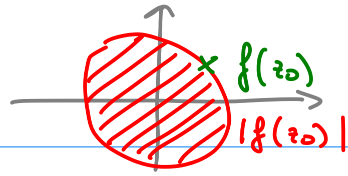
In particular, if $\overline{U}$ is bounded and f is a function continuous on $\overline{U}$ and holomorphic on U, then

$$\underbrace{\max_{z \in \overline{U}} |f(z)|}_{\text{exists because } \overline{U} \text{ compact}} = \max_{z \in \overline{U}-U} |f(z)| .$$

<u>Example</u> - If $f \in D_1(0)$ then for $0<r<1$, we get

$$\max_{|z| \leq r} |f(z)| = \max_{|z|=r} |f(z)| .$$

<u>Proof</u> - Suppose z_0 exists. Then for any



$\delta > 0$, we get

$$\{ f(z) \mid |z - z_0| < \delta \} \subset \{ w \in \mathbb{C} \mid |w| \leq |f(z_0)| \}$$

which does not contain a disc centered at $f(z_0)$, and this would contradict the open-image theorem, unless f is constant.

For the second statement, note that we know that there is $z_0 \in \overline{U}$ s.t. $\forall z \in \overline{U}$, $|f(z)| \leq |f(z_0)|$ (because $|f|$ is continuous and $\overline{U}$ compact), and the first statement shows that $z_0 \in U$ is not possible.

$\square$