Chapter VI

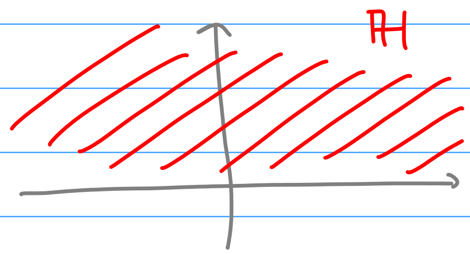
A long example

Eta Theta Zeta

We will show in this chapter some applications of the general theory we have developed. This concern some of the most important functions in mathematics.

1 - Eta, Theta, Zeta

We will present three functions:



(1) [Dekind's Eta function]

Let $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \} \subset \mathbb{C}$ (open, connected)

and

$$\eta : \mathbb{H} \longrightarrow \mathbb{C}$$

$$z \longmapsto e^{\frac{i\pi z}{12}} \underbrace{\prod_{n \geq 1} \left(1 - e^{2i\pi n z}\right)}_{\displaystyle \lim_{N \to \infty} \prod_{n=1}^{N} \left(1 - e^{2i\pi n z}\right)}$$

As we will see, $\eta \in \mathcal{H}(\mathbb{H})$.

(2) [Jacobi's Theta function]

This was already seen in a previous chapter:

$$\theta : \mathbb{H} \longrightarrow \mathbb{C}$$
$$z \longmapsto \sum_{n \in \mathbb{Z}} e^{2i\pi n^2 z}$$
$$= 1 + 2 \sum_{n=1}^{+\infty} e^{2i\pi n^2 z}$$

We have $\theta \in \mathcal{H}(\mathbb{H})$.

(3) [Riemann's Zeta function]

For $s = \sigma + it \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > 1$

let $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$. Then ζ is

holomorphic on this open (connected set).

One can show that η, θ cannot be continued to holomorphic (or even meromorphic) functions on a connected set containing $\mathbb{H}$. On the other hand, one can [and we will] show:

Theorem. The zeta function admits meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s=1$ with residue 1. In fact we then have

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

("functional equation")

The main reason the zeta function is studied is however because of its link with prime numbers.

This is given by

Theorem ("Euler"). For $\mathrm{Re}(s) > 1$, we have

$$\zeta(s) = \underbrace{\prod_{\substack{p \\ \text{prime}}} (1 - p^{-s})^{-1}}_{\displaystyle \lim_{X\to+\infty} \prod_{\substack{2 \le p \le X \\ p \text{ prime}}} (1-p^{-s})^{-1}}$$

Before saying more about this, let's digress a bit about infinite products.

2. Infinite products

Definition. Let $(a_n)_{n\geq 0}$ be a sequence of

complex numbers. The infinite product $\prod_{n\geq 1}(1+a_n)$ is defined as the limit, if it exists (in which case we say that the limit is the product of the $1+a_n$ / that the product converges) of the sequence
$$P_N = \prod_{n=1}^{N} (1+a_n).$$

The basic properties we need are summarized in the following result:

Proposition.

(1) If $\sum |a_n|$ converges, then $\prod (1+|a_n|)$ and $\prod (1+a_n)$ converge; the product is zero if and only if some a_n is equal to -1. (We then say that the product converges absolutely.)

(2) If $U \subset \mathbb{C}$ is open, $a_n \in \mathcal{H}(U)$ and $\sum a_n(z)$ converges absolutely uniformly on compact subsets, then
$$f(z) = \prod_{n=1}^{+\infty} (1 + a_n(z))$$
is holomorphic on U.

Proof. (1) Note first that

$$\prod_{n=1}^{N} (1 + |a_n|) \leq \prod_{n=1}^{N} \exp(|a_n|)$$

$$(1+|z| \leq e^{|z|})$$

$$= \exp\left(\sum_{n=1}^{N} |a_n|\right)$$

so the assumption implies that the non-decreasing sequence $\prod_{n=1}^{N}(1+|a_n|)$ is bounded from above, and therefore converges.

Next, note that

$$\prod_{n=1}^{N+1}(1+a_n) - \prod_{n=1}^{N}(1+a_n)$$

$$= \prod_{n=1}^{N}(1+a_n)\,(1+a_{N+1}-1)$$

so

$$\left|\prod_{n=1}^{N+1}(1+a_n) - \prod_{n=1}^{N}(1+a_n)\right|$$

$$\leq |a_{N+1}| \prod_{n=1}^{N}(1+|a_n|)$$

$$= \prod_{n=1}^{N+1}(1+|a_n|) - \prod_{n=1}^{N}(1+|a_n|)$$

so for $M \geq N \geq 1$

$$|P_M - \underbrace{P_N}_{= \prod_{n=1}^{N}(1+a_n)}| \leq \tilde{P}_M - \underbrace{\tilde{P}_N}_{= \prod_{n=1}^{N}(1+|a_n|)}$$

which therefore implies that $(\mathcal{P}_N)_{N\geq 1}$ is a Cauchy sequence, and therefore converges.

Also if $a_n \neq -1$, then $\sum \left|\frac{a_n}{1+a_n}\right| < +\infty$ so the product

$$\prod_{n=1}^{N}\left(1-\frac{a_n}{1+a_n}\right) = \prod_{n=1}^{N} \frac{1}{1+a_n} = \frac{1}{\prod_{n=1}^{N}(1+a_n)} \text{ converges,}$$

which means that $\prod(1+a_n) \neq 0$.

(2) For $K \subset U$ compact and $z \in K$, we get

$$|\mathcal{P}_M(z) - \mathcal{P}_N(z)| \leq \widetilde{\mathcal{P}}_M(z) - \widetilde{\mathcal{P}}_N(z)$$

so letting $M \to \infty$, we get

$$|f(z) - \mathcal{P}_N(z)| \leq g(z) - \widetilde{\mathcal{P}}_N(z)$$

with $g(z) = \prod_{n\geq 1}(1+|a_n(z)|) \leq \exp\left(\sum_{n=1}^{+\infty}|a_n(z)|\right)$

and so $\mathcal{P}_N(z)$ converges uniformly to $f(z)$ on K. Since each $\prod_{n=1}^{N}(1+a_n)$ is in $\mathcal{H}(U)$, the convergence theorem implies $f \in \mathcal{H}(U)$.

$\square$

3 - The convergence of the η and Euler products

Proposition - (1) $\eta \in \mathcal{H}(\mathbb{H})$

(2) The <u>Euler product</u> $\prod_p (1-p^{-s})^{-1}$ converges to a holomorphic function for $Re(s)>1$ and it is equal to $\zeta(s)$ for all such s.

<u>Proof.</u> (1) Let $a_n(z) = e^{2i\pi n z}$. Then $|a_n(z)| = e^{-2\pi n Im(z)}$ for $z \in \mathbb{C}$. Since

$$\sum |a_n(z)| = \sum \alpha(z)^n$$

with $\alpha(z) = e^{-2\pi Im(z)} \in]0,1[$ for $z \in \mathbb{H}$, and moreover $|\alpha(z)| \leq e^{-2\pi y_0}$ if $Im(z) \geq y_0$, we get from the proposition of the previous section that $\eta \in \mathcal{H}(\mathbb{H})$.

(2) Observe that if p is prime, then for $s = \sigma + it$, we have $|p^s| = |e^{s \log p}| = e^{\sigma \log p} = p^\sigma$.

Then $\dfrac{1}{1-p^{-s}} = 1 + a_p(s)$, where

$$a_p(s) = \frac{p^{-s}}{1-p^{-s}}$$

is holomorphic on $\mathbb{C}$, and

$$|a_p(s)| \leq \frac{p^{-\sigma}}{1-p^{-\sigma}}$$

$\Big(|1-p^{-s}| \geq 1 - |p^{-s}| = 1 - p^{-\sigma}\Big)$

defines a uniformly convergent series on the set

$$\{ s \in \mathbb{C} \mid \sigma \geqslant 1+\delta \}$$

for any $\delta > 0$ (because

$$|a_p(s)| \leqslant \frac{1}{1-2^{-\sigma}} \, p^{-\sigma}$$

and $\displaystyle\sum_p p^{-\sigma} \leqslant \sum_{n=1}^{+\infty} n^{-1-\delta}$ in this set).

So the product does converge to a holomorphic function on $\{ \mathrm{Re}(s) > 1 \}$.

Now pick $X \geqslant 1$ and $\sigma > 1$ real. Then

$$P_X(\sigma) = \prod_{\substack{2 \leq p \leq X \\ p \text{ prime}}} \left(1 - \frac{1}{p^{\sigma}}\right)^{-1}$$

$$= \prod_{p \leq X} \left(1 + p^{-\sigma} + \dots + p^{-n\sigma} + \dots\right)$$

$$= \sum_{n \in S_X} n^{-\sigma}$$

where S_X is the set of all integers which have only prime factors $p \leqslant X$ (we use here the fundamental theorem of arithmetic: the uniqueness of expansion in primes ensures that the coefficient of $n^{-\sigma}$ is 1). So

$$\sum_{n \leq x} \frac{1}{n^\sigma} \leq P_x(\sigma) \leq \sum_{n \in S_x} \frac{1}{n^\sigma} \leq \sum_{n \geq 1} \frac{1}{n^\sigma}$$

and letting $x \to +\infty$, we get

$$\prod (1 - p^{-\sigma})^{-1} = \zeta(\sigma).$$

Now both sides are values at $\sigma > 1$ of holomorphic functions! So by analytic continuation, we must have

$$\zeta(s) = \prod_p \frac{1}{1 - 1/p^s}$$

for <u>all</u> s s.t. $\mathrm{Re}(s) > 1$.

$\square$

4. <u>Analytic continuation and functional equation</u>

We are going to connect ζ and θ: remembering that $\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt$ for $\mathrm{Re}(s) > 1$

we get

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^{+\infty} e^{-t} \left(\frac{t}{\pi n^2}\right)^{s/2} \frac{dt}{t}$$

$u = \dfrac{t}{\pi n^2}$; $dt = \dfrac{du}{\pi n^2}$

$$= \int_0^{+\infty} e^{-\pi n^2 u} u^{s/2} \frac{du}{u}$$

$\dfrac{dt}{t} = \dfrac{du}{u}$

for $n \geq 1$, and then

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{+\infty} \left(\sum_{n=1}^{+\infty} e^{-\pi n^2 u} \right) u^{s/2} \frac{du}{u}$$

$$= \frac{1}{2} \int_0^{+\infty} \left(\theta\left(i \frac{u}{2}\right) - 1 \right) u^{s/2} \frac{du}{u}$$

$\left(\text{since } \theta(iy) = 1 + 2 \sum_{n \geq 1} e^{2i\pi n^2 (iy)} = 1 + 2 \sum e^{-2\pi n^2 y} \right)$

where exchanging $\int$ and $\sum$ is fairly simple.

Now we use:

Proposition. For $y > 0$ we have

$$\theta\left(\frac{iy}{2}\right) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / y}$$

$$= \frac{1}{\sqrt{y}} \theta\left(\frac{i}{2y}\right).$$

This leads us to split the integral from 0 to 1 and 1 to $+\infty$:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} \left(\int_0^1 \cdots + \int_1^{\infty} \cdots \right)$$

Then

$$\int_0^1 \left(\theta\left(i \frac{u}{2}\right) - 1 \right) u^{s/2} \frac{du}{u}$$

$\left(u = \frac{1}{y} ; \quad \frac{du}{u} = -\frac{dy}{y} \right)$

$$= \int_1^{+\infty} \theta\left(\frac{i}{2y}\right) y^{\frac{1-s}{2}} \frac{dy}{y} - \int_0^1 u^{\frac{s}{2}-1} du$$

$$= \int_1^{\infty} \left(\theta\left(\frac{i}{2y}\right) - 1\right) y^{\frac{1-s}{2}} \frac{dy}{y}$$
$$- \int_0^1 u^{\frac{s}{2}-1} du + \int_1^{\infty} y^{\frac{1-s}{2}} \frac{dy}{y}$$

$$= \int_1^{+\infty} \left(\theta\left(\frac{i}{2y}\right) - 1\right) y^{\frac{1-s}{2}} \frac{dy}{y}$$
$$- \frac{2}{s} - \frac{2}{1-s}$$

and so

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}\right) = \left(\frac{1}{s-1} - \frac{1}{s}\right)$$
$$+ \int_1^{\infty} \left(\theta\left(\frac{iu}{2}\right) - 1\right) \left(u^{s/2} + u^{\frac{1-s}{2}}\right) \frac{du}{u}.$$

One can easily check that the integral defines a function holomorphic on $\mathbb{C}$ (because

$$\theta(iy) - 1 = 2 \sum_{n \geq 1} e^{-2\pi n^2 y}$$

is exponentially small for y large) so

$$\Lambda(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta\left(\frac{s}{2}\right)$$

is meromorphic on $\mathbb{C}$, and invariant under $s \longmapsto 1 - s$. The function Λ has poles of order 1 at $s = 0$

(with residue -1) and $s=1$ (with residue 1).

But $\Gamma(\frac{s}{2})$ also has a pole at $s=0$ of order 1 so $\zeta \in \mathcal{M}(\mathbb{C})$ has no pole at $s=0$;

since $\Gamma(\frac{s}{2})$ has no pole at $s=1$, it follows that ζ has a simple pole at $s=1$ with residue $\lim_{s\to 1}(s-1)\zeta(s) = 1$ (because e.g.

$$\pi^{-1/2}\,\Gamma\left(\tfrac{1}{2}\right)\operatorname{res}_1 \zeta(s) = \lim_{s\to 1}(s-1)\Lambda(s) = 1$$

and $\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} e^{-t}\sqrt{t}\,\frac{dt}{t} = 2\int_0^{+\infty} e^{-u^2}\,du$
$= \sqrt{\pi}$).

How does one prove the proposition? It is a special case of the "Poisson summation formula":

(IV.2.4)

Theorem - If $f:\mathbb{R}\to\mathbb{C}$ tends to 0 fast enough at $\pm\infty$ then for $x\in\mathbb{R}$

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{h\in\mathbb{Z}} \int_{-\infty}^{+\infty} f(t)\, e^{-2i\pi h t}\,dt.$$

(Sketch of proof: define $\varphi(x) = \sum_{n\in\mathbb{Z}} f(n+x)$;

then note that φ is 1-periodic and check that the Fourier coefficients of f are

$$\int_0^1 \varphi(x)\, e^{-2i\pi h x}\, dx = \int_{-\infty}^{+\infty} f(t)\, e^{-2i\pi h t}\, dt$$

then check that the Fourier formula

$$\varphi(0) = \sum_{h \in \mathbb{Z}} \hat{\varphi}(h)$$

is valid.)

5 – How to count prime numbers

Historically, much of the basic theory of complex analysis was motivated by Riemann's suggestion to use it as a tool to prove a result which had been guessed by Gauss:

Theorem – (The "Prime Number Theorem"; Hadamard, de la Vallée Poussin, 1896)

Let $\pi(x)$ = nb. of primes $p \leq x$ $(x \geq 2)$.

Then $\pi(x) \sim \dfrac{x}{\log x}$ as $x \longrightarrow +\infty$

$$\left(\lim_{x \to \infty} \frac{\pi(x)}{x / \log x} = 1 \right)$$

We conclude this example by sketching Riemann's approach to this problem.

<u>Step 1</u>. To get a <u>sum</u> instead of a product, consider the logarithmic derivative

$$\frac{\zeta'}{\zeta}(s) = -\sum_p \frac{-(p^{-s})'}{1-p^{-s}} = \sum_p \frac{-(\log p)\, p^{-s}}{1-p^{-s}}$$

for $\mathrm{Re}(s) > 1$. Since $|p^{-s}| < 1$ then we get

$$-\frac{\zeta'}{\zeta}(s) = \sum_p (\log p)\left(p^{-s} + p^{-2s} + \dots\right)$$
$$= \sum_{n\geq 1} \Lambda(n)\, n^{-s}$$

where Λ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for } \begin{cases}\text{some prime } p \\ \text{some } m \geq 1\end{cases} \\ 0 & \text{otherwise.}\end{cases}$$

<u>Step 2</u>. Because $\log x$ increases slowly it is easy to check that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1 \iff \lim_{x\to\infty} \frac{\psi(x)}{x} = 1$$

where

$$\psi(x) = \sum_{n\leq x} \Lambda(n).$$

$$\Bigg(= \underbrace{\sum_{p\leq x} \log p}_{\approx (\log x)\,\pi(x)} + \underbrace{\sum_{p\leq \sqrt{x}} \log p}_{\leq (\log x)\sqrt{x}} + \dots \Bigg)$$

Step 3. We express $\psi(x)$ by complex integration:

Lemma. ("Perron formula")

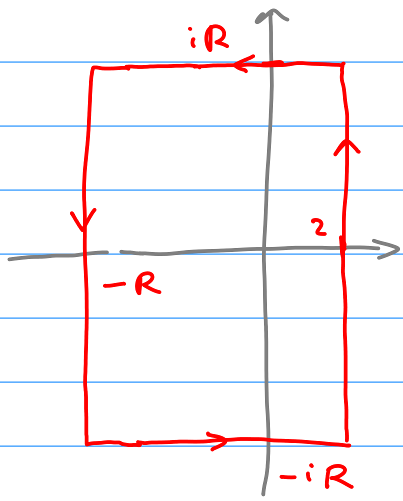
For $a > 0$ in $\mathbb{R}$ $(a \notin \mathbb{Z})$ we have

$$\lim_{R \to +\infty} \frac{1}{2i\pi} \int_{2-iR}^{2+iR} a^s \, \frac{ds}{s} = \begin{cases} 1 & \text{if } a > 1, \\ 0 & \text{if } a < 1 \end{cases}$$

Sketch of proof. If $a > 1$, consider

$$I_R = \int_{\gamma_R} a^s \, \frac{ds}{s}$$

where γ_R is the curve



The Residue Formula gives

$$I_R = 2i\pi \operatorname{res}_{s=0} a^s/s = 2i\pi$$

and on the other hand the integral on the vertical segment on the right is

$$-i \int_{-R}^{R} a^{-R} \frac{dt}{-R+it}$$

$s = \sigma + it$

$ds = i\,dt$

with $\quad |-| \leq a^{-R} \int_{-R}^{R} \frac{dt}{\sqrt{R^2+t^2}}$

$$\xrightarrow[R \to +\infty]{} 0 \quad (\text{since } a > 1)$$

and similarly the integrals on the horizontal segments tend to 0.

$\square$

Using this we get

$$\sum_{n \leq x} \Lambda(n) = \sum_{n \geq 1} \Lambda(n) \lim_{R \to \infty} \frac{1}{2i\pi} \int_{2-iR}^{2+iR} \left(\frac{x}{n}\right)^s \frac{ds}{s}$$

$$= \lim_{R \to \infty} \frac{1}{2i\pi} \int_{2-iR}^{2+iR} \underbrace{\left(\sum_{n \geq 1} \Lambda(n) n^{-s}\right)}_{-\frac{\zeta'}{\zeta}(s)} x^s \frac{ds}{s}$$

So to count primes we should study

$$\frac{1}{2i\pi} \int_{2-iR}^{2+iR} - \frac{\zeta'}{\zeta}(s) \, x^s \, \frac{ds}{s}.$$

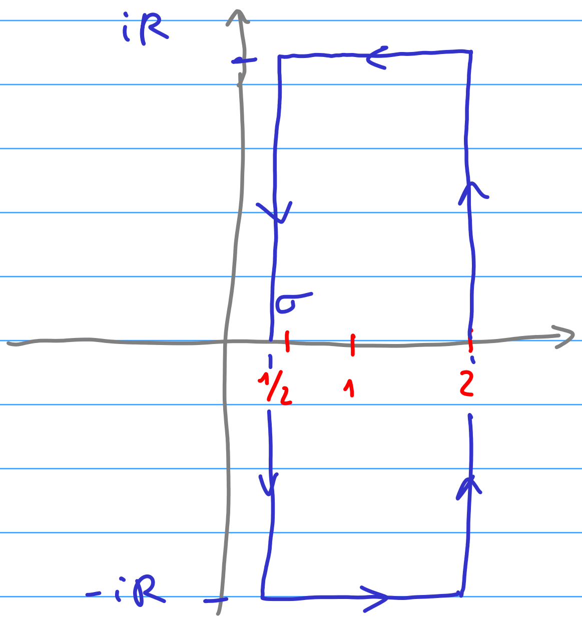
Note that $-\frac{\zeta'}{\zeta}$ is meromorphic on $\mathbb{C}$ and has:

(1) simple poles with residue at all $\rho \in \mathbb{C}$ s.t. $\zeta(\rho) = 0$, with residue <u>minus</u> the multiplicity;

(2) simple pole with residue 1 at $s = 1$.

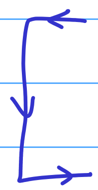
So if we proceed by enclosing (some of) these we get

$$\frac{1}{2i\pi}\int_{\gamma_R} -\frac{\zeta'}{\zeta}(s)\, x^s \frac{ds}{s} = -\sum_{\substack{|\mathrm{Im}\,\rho| \le R \\ \sigma \le \mathrm{Re}(\rho) \le 2}} \mathrm{ord}_\rho(\zeta)\, \frac{x^\rho}{\rho} + x$$



(if we choose R so that no zero has imaginary part ρ)

Now how to choose σ?

We need to show that the remaining part of the integral  is small.

On the vertical segment with $\mathrm{Re}(s) = \sigma$, we cannot easily improve the bound

$$\left| \frac{\zeta'}{\zeta}(s)\, x^s \frac{1}{s} \right| \le x^\sigma \left| \frac{\zeta'}{\zeta}(s) \frac{1}{s} \right|$$

so it is better to take σ as small as possible.

On the other hand, to see that $\sum_{n \le x} \Lambda(n)$ is close to x, we need the zeros ρ to also have real part as small as possible. Because of the functional equation, if there is a zero ρ, then $1-\rho$ is another one. So if there are zeros, then there are some with $Re(\rho) \geq \frac{1}{2}$.

The Riemann Hypothesis - If $0 \le Re(\rho) \le 1$ and $\zeta(\rho) = 0$ then $Re(\rho) = \frac{1}{2}$.

"Theorem" (Riemann) - If RH is true, then

$$\sum_{n \le x} \Lambda(n) = x + O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

for all $\varepsilon > 0$ (i.e. for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ s.t.

$$\left| \sum_{n \le x} \Lambda(n) - x \right| \le C_\varepsilon \, x^{\frac{1}{2}+\varepsilon}$$

for all $x \geq 1$)

This remains unproved, but in 1896, Hadamard and de la Vallée Poussin succeeded in avoiding this hypothesis by proving:

<u>Theorem</u>. If $\mathrm{Re}(s) = 1$ then $\zeta(s) \neq 0$.

Then they showed that this suffices at least to prove the Prime Number Theorem.

<u>Sketch of Proof</u>.

Observe the inequality

$$3 + 4\cos(\theta) + \cos(2\theta) = 2(1+\cos(\theta))^2 \geq 0$$

for all $\theta \in \mathbb{R}$, and from this note that

$$|\zeta(s)| = \exp\left(\sum_p \sum_{k \geq 1} \frac{\cos(kt \log p)}{p^s}\right)$$

$(\mathrm{Re}(s) > 1)$ so

$$|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)| \geq 1.$$

If $\zeta(1+it) = 0$, then (because $4 \geq 3$) we get $\zeta(\sigma+2it) \xrightarrow[\sigma \to 1]{} +\infty$ (ζ has a simple pole at $s=1$), which is impossible since ζ has only one pole, at $s = 1$.

$\square$

6 - Some nice formulas

(1) [Euler] For $k \geq 1$, integer

$$\zeta(2k) = \pi^{2k} \alpha_k$$

$\left(= \sum_{n=1}^{+\infty} \frac{1}{n^{2k}}\right)$ where α_k is a non-zero rational number, precisely

$$\alpha_k = \frac{2^{2k-1}}{(2k)!} B_k$$

Bernoulli numbers

with B_k defined by the power series expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{+\infty} (-1)^{k+1} B_k \frac{z^{2k}}{(2k)!}$$

Open Question: show that $\zeta(3)$ is not of the form $\pi^3 \alpha_3$ with $\alpha_3 \in \mathbb{Q}$ ...

(2) [Jacobi; Ramanujan; Deligne]

We defined $\eta(z) = e^{\frac{i\pi z}{12}} \prod_{n\geq 1} \left(1 - e^{2i\pi n z}\right)$

for $\operatorname{Im}(z) > 0$. In applications, one encounters

also
$$\Delta(z) = \eta(z)^{24}$$
$$= e^{2i\pi z} \prod_{n=1}^{+\infty} \left(1 - e^{2i\pi n z}\right)^{24}.$$

Jacobi proved that
$$\Delta\left(-\frac{1}{z}\right) = z^{12}\,\Delta(z)$$
for $\mathrm{Im}(z) > 0$.

Ramanujan looked at the coefficients $\tau(n)$ in the expansion $\Delta(z) = \sum_{n=1}^{+\infty} \tau(n)\, e^{2i\pi n z}$; it is easy to see that $\tau(n) \in \mathbb{Z}$ for all n.

Ramanujan conjectured:

(1) $\tau(mn) = \tau(m)\,\tau(n)$ if m is coprime to n

(e.g. $\tau(15) = \tau(3)\,\tau(5)$)

This was proved by Mordell (1920's)

(2) If p is prime, then
$$|\tau(p)| < p^{11/2}$$

This was proved by Deligne (1974), by it is an extraordinarily deep / difficult result!