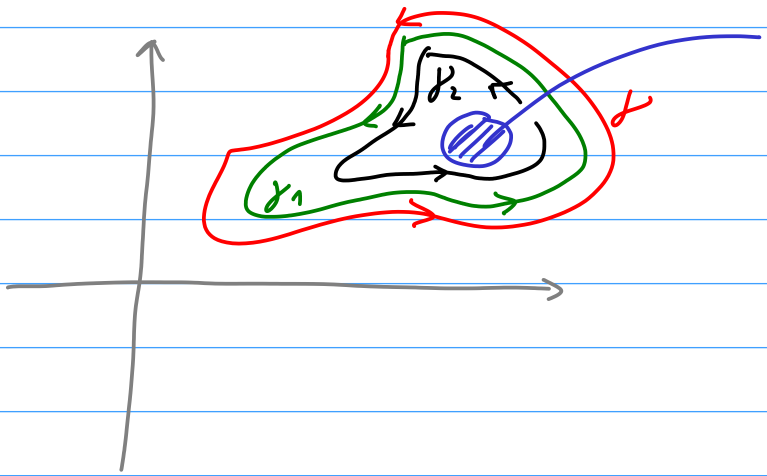
Chapter VII

Homotopy and applications

The best way to understand and generalize the residue integral formula is to use the following principle: The value of a line integral $\int_\gamma f(z)\,dz$, for a closed $\gamma \subset U$ and $f \in \mathcal{H}(U)$ does not change if we "deform" γ while remaining in U.



$$\int_\gamma f = \int_{\gamma_1} f = \int_{\gamma_2} f$$

1 - Homotopy

Two curves γ_1, γ_2 which can be "deformed" into each other are called <u>homotopic</u>. The precise definition is as follows.

Definition. $U \subset \mathbb{C}$ open

Let $\left.\begin{array}{l}\gamma_0 : [a,b] \longrightarrow U \\ \gamma_1 : [a,b] \longrightarrow U\end{array}\right\}$ smooth, closed curves in U

We say that γ_0 is homotopic to γ_1 in U if there exists $F : [o,b] \times [0,1] \longrightarrow U$ continuous such that $\left\{\begin{array}{l} F(t,0) = \gamma_0(t), \\ F(t,1) = \gamma_1(t) \end{array}\right.$

and $\left.\left\{\begin{array}{l}\gamma_u(t) = F(t,u) \text{ is a smooth curve} \\ F(0,u) = F(1,u)\end{array}\right.\right\}$ for $u \in [0,1]$.

If γ_0, γ_1 are not closed but have the same endpoints $(\gamma_0(a) = \gamma_1(a),\ \gamma_0(b) = \gamma_1(b))$ they are homotopic (in U) with endpoints fixed if there is an $F : [a,b] \times [0,1] \longrightarrow U$ continuous with $\left\{\begin{array}{l} F(t,0) = \gamma_0(t) \\ F(t,1) = \gamma_1(t)\end{array}\right.$

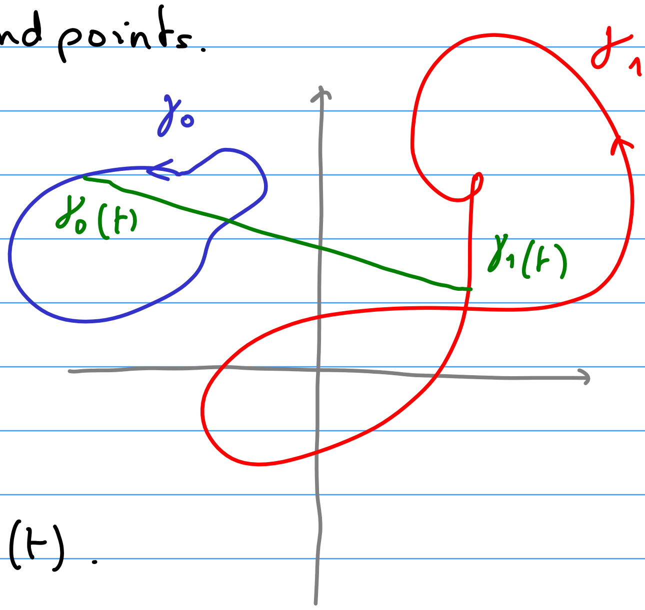
and $F(0,u) = \gamma_0(a)$, $F(1,u) = \gamma_1(b)$

for all u.

Examples – (1) Let $U = \mathbb{C}$. Then we claim that

(i) Any closed curves γ_1, γ_2 are homotopic (in $\mathbb{C}$)

(ii) Any curves γ_0, γ_1 with the same endpoints are homotopic with fixed endpoints.



Both facts follow from the same definition de $F(t,u)$:

$$F(t,u) = (1-u)\gamma_0(t) + u\,\gamma_1(t).$$

Indeed, this defines $F : [0,b] \longrightarrow [0,1]$, which is continuous (as combination of continuous functions) and

$$\begin{cases} F(t,0) = 1\cdot\gamma_0(t) + 0\cdot\gamma_1(t) = \gamma_0(t) \\ F(t,1) = 0\cdot\gamma_0(t) + 1\cdot\gamma_1(t) = \gamma_1(t). \end{cases}$$

[Geometrically, as <u>u varies</u> for <u>fixed</u> t, the value of $F(t,u)$ runs over the line segment in

[between $\gamma_0(t)$ and $\gamma_1(t)$.]

If the curves are closed then

$$F(0,u) = (1-u)\gamma_0(0) + u\gamma_1(0)$$

$$= (1-u)\gamma_0(1) + u\gamma_1(1)$$

$$= F(1,u)$$

so the intermediate curves are closed.

If the endpoints are fixed, then

$$F(0,u) = (1-u)\gamma_0(0) + u\gamma_1(0)$$

$$= (1-u)\gamma_0(0) + u\gamma_0(0)$$

$$= \gamma_0(0) \quad (= \gamma_1(0))$$

$$F(1,u) = \gamma_0(1) \quad (= \gamma_1(1))$$

so the homotopy is with fixed endpoints.

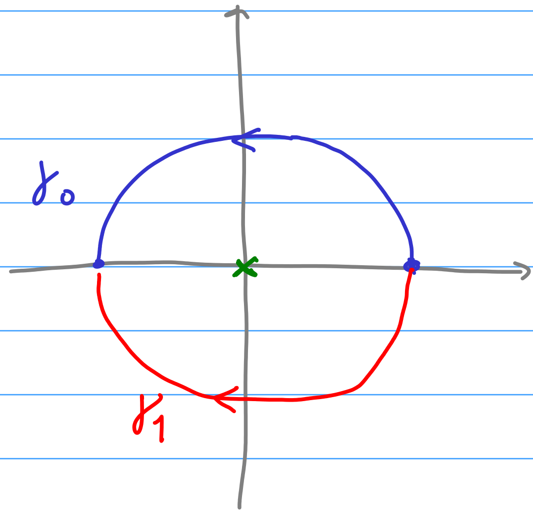
(2) More generally, if $U \subset \mathbb{C}$ is a convex open set, then we can also apply the same formula, and the properties (i) and (ii) hold in U also.

(3) Let $U = \mathbb{C} - \{0\} = \mathbb{C}^*$. Then the curves defined by

$$\begin{cases} \gamma_0(t) = e^{it}, & 0 \le t \le \pi \\ \gamma_1(t) = e^{-it}, & 0 \le t \le \pi \end{cases}$$

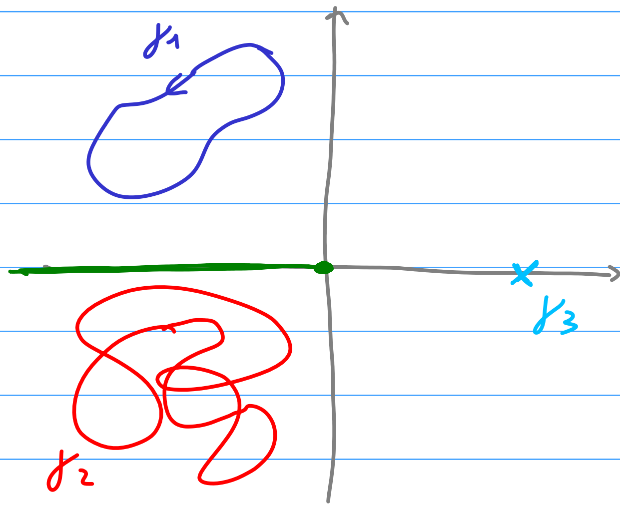
are <u>not</u> homotopic [with fixed endpoints] in $\mathbb{C}^*$.

This is intuitively clear, and we will see a simple way to confirm it later.



(4) Let $U = \mathbb{C} -]-\infty, 0]$.

Although U is not convex (for instance the segment from $-1-i$ to $-1+i$ is not contained in U)



the properties (i) and (ii) of Examples (1) and (2) apply.

For instance, given closed curves γ_1 and γ_2,

we define

$$F(t,u) = \begin{cases} 1 + (1-2u)(\gamma_0(t) - 1), & 0 \le u \le \frac{1}{2} \\ 1 + 2\left(u - \frac{1}{2}\right)(\gamma_1(t) - 1) & \frac{1}{2} < u \le 1 \end{cases}$$

Then F is continuous [because $F(t, \frac{1}{2}) = 1$ for all t, which is $\lim\limits_{\substack{t \to 1/2 \\ t > 1/2}} F(t, \frac{1}{2})$ also]. It is a map from $[a,b] \times [0,1]$ to U. Indeed suppose that $a \le t \le b$, $0 \le u \le \frac{1}{2}$; then $F(t,u) \notin U$ means that

$$1 + (1-2u)(\gamma_0(t) - 1) \le 0$$

(meaning that it is a real number and is ≤ 0), which means

$$(1-2u)(\gamma_0(t) - 1) \le -1$$

$$\iff$$

$$\gamma_0(t) \le 1 - \frac{1}{1-2u} = -\frac{2u}{1-2u} \le 0$$

$\begin{cases} 2u \geq 0 \\ 1-2u \geq 0 \end{cases}$

and this is never the case because $\gamma_0(t) \in U$ for all t. Similarly for $\frac{1}{2} < u \le 1$.

Next, we note that

$$\begin{cases} F(t,0) = 1 + 1\cdot(\gamma_0(t) - 1) = \gamma_0(t) \\ F(t,1) = 1 + 2\cdot\frac{1}{2}\cdot(\gamma_1(t) - 1) = \gamma_1(t) \end{cases}$$

and $F(0,u) = 1 + (1-2u)(\gamma_0(0) - 1)$

$$= 1 + (1-2u)(\gamma_0(1) - 1)$$

$$= F(1,u)$$

for all u, so F is an homotopy of closed curves.

For homotopy with fixed endpoints, write

$$\begin{cases} \gamma_0(t) = r_0(t)\, e^{i\theta_0(t)} \\ \gamma_1(t) = r_1(t)\, e^{i\theta_1(t)} \end{cases}$$

with $\left.\begin{array}{l} r_0, r_1 : [a,b] \longrightarrow]0,+\infty[\\ \theta_0, \theta_1 : [a,b] \longrightarrow]-\pi,\pi[\end{array}\right\}$ continuous

We can then define

$$F(t,u) = \left\{(1-u)r_0(t) + u\, r_1(t)\right\} e^{i\left\{(1-u)\theta_0(t) + u\,\theta_1(t)\right\}}$$

to get a homotopy with fixed endpoints.

2 - The homotopy theorem

We can now state the precise version of the statement about the invariance of line integrals under continuous deformation.

Theorem - [III.5.1] "Homotopy Theorem" $U \subset \mathbb{C}$ open, γ_0, γ_1 curves in U such that either

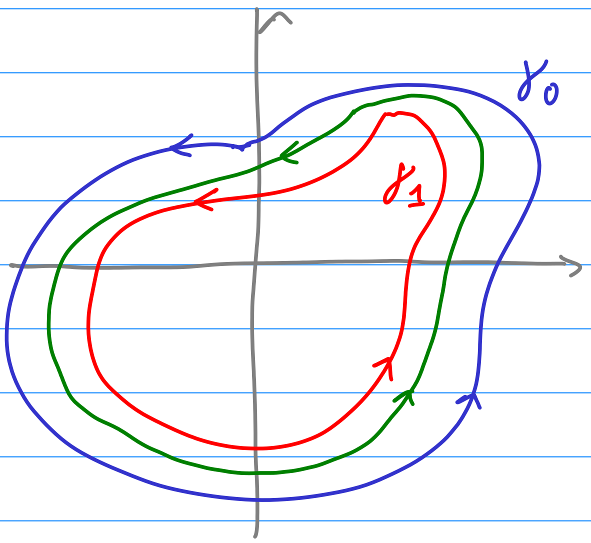
(i) γ_0 and γ_1 are closed and homotopic;

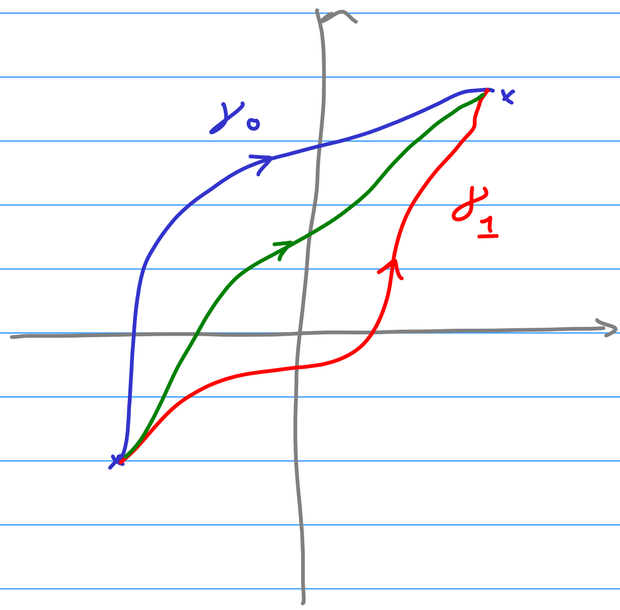
or

(ii) γ_0 and γ_1 have the same endpoints and are homotopic with fixed endpoints.

For $f \in \mathcal{H}(U)$, we have

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$



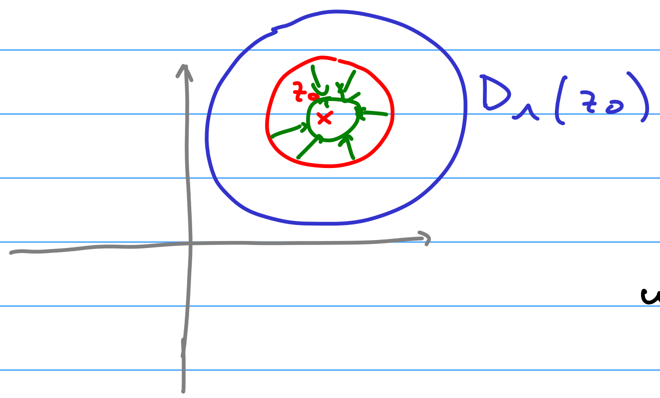


Example. (1) Consider $U = D_r(z_0)$, $r > 0$.

Intuitively, the circle $\gamma = C_s(z_0)$, $0 < s < r$ can be deformed by "dilation" into the circle $C_0(z_0) = \{z_0\}$. So

$$\int_{C_r(z_0)} f(z)\,dz = \int_{\{z_0\}} f(z)\,dz$$

$$= 0$$



if $f \in \mathcal{H}(D_r(z_0))$, which is Cauchy's Theorem.

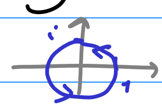
(2) Consider $U = \mathbb{C}^*$ and the curves

$$\gamma_1(t) = e^{it}, \quad \gamma_2(t) = e^{-it}, \quad 0 \leq t \leq \pi.$$

These are not homotopic with fixed endpoints in U because if they were, it would follow that

$$\int_{\gamma_1} \frac{1}{z}\,dz = \int_{\gamma_2} \frac{1}{z}\,dz$$

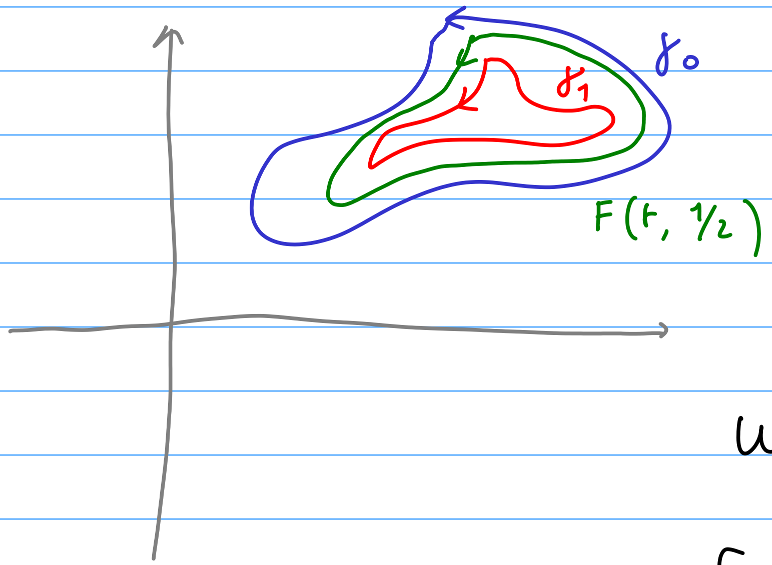
whereas $\displaystyle \int_{\gamma_1} \frac{1}{z}\,dz - \int_{\gamma_2} \frac{1}{z}\,dz = \int \frac{1}{z}\,dz = 2i\pi$



(cf. Chapter II, p° 22).

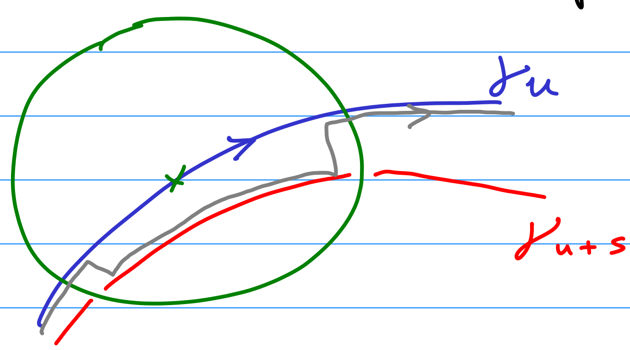
3. Proof of the Theorem

Let us first understand the idea in the case of closed curves [the book contains the proof with fixed endpoints].



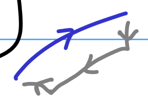
We denote by γ_u the closed curve $t \longmapsto F(t, u)$.

We first show that for any $u \in [0, 1[$, we can find $\varepsilon > 0$ such that $\int_{\gamma_u} f(z)\,dz = \int_{\gamma_{u+s}} f(z)\,dz$ if $0 \leq s < \varepsilon$. (This corresponds to a "small" deformation.) To see why this should be true, we look around a point of γ_u, in a small disk contained in U.



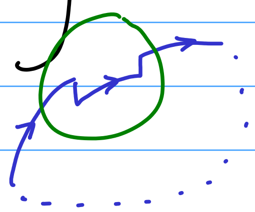
For $s > 0$ very small, a small piece of γ_{u+s} is in the same disk. We can "switch tracks" in the disc and use Cauchy's Theorem in the disc:

$$\int f(z)\,dz = 0$$



because f has a primitive in the small disc, so

$$\int_{\gamma_u} f(z)\,dz = \int f(z)\,dz.$$



Doing this over a (finite) sequence of small discs, we can go from u to $u+s$ for s small enough.

Intuitively this will be enough because everything will happen within a compact set so only a finite number of "steps" will be needed.

Now we come to the details.

Let $K \subset U$ be the image of F; this is a compact set (because F is continuous and $[a,b] \times [0,1] \subset \mathbb{R}^2$ is compact).

<u>Lemma</u>. There exists $\delta > 0$ such that for all $z \in K$, the disc $D_\delta(z)$ is contained in U.

Proof. Assume this is not the case: then, for any integer $n \geqslant 1$, we find $z_n \in K$ and $w_n \notin U$ such that $|z_n - w_n| \leqslant \frac{1}{n}$. Since K is compact, there is a convergent subsequence $(z_{n_k})_{k \geqslant 1}$, with limit z. Then $w_{n_k} \longrightarrow z$ also. But $z_{n_k} \in K$ so $z \in K$ (K is closed) whereas $w_{n_k} \in \mathbb{C} - U$ which is also closed, so we would get $z \in \mathbb{C} - U$, and this is a contradiction.

$\square$

Next, since $F : [a,b] \times [0,1] \longrightarrow U$ is uniformly continuous, we can find an integer $N \geqslant 1$ such that, writing

$$x_{m,n} = \Big(\underbrace{a + m \frac{b-a}{N}}_{t_m} , \underbrace{\frac{n}{N}}_{u_n} \Big)$$

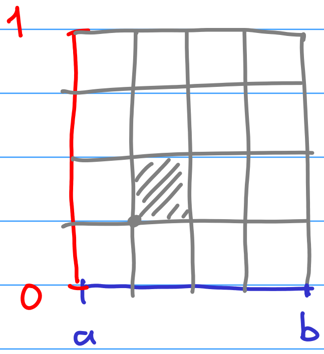
$$0 \leqslant m \leqslant N, \quad 0 \leqslant n \leqslant N$$

we have

$$|F(t,u) - F(x_{m,n})| < \delta$$

if $t_m \leq t \leq t_{m+1}$, $u_n \leq u \leq u_{n+1}$.

(In other words, the image by F of each small square



$$[t_m, t_{m+1}] \times [u_n, u_{n+1}]$$

is contained in the disc of radius δ around $F(x_{m,n})$.)

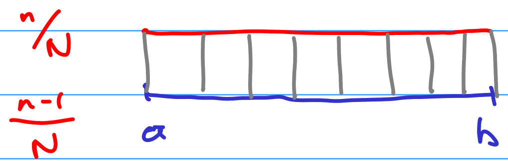
We now check, by induction on n, $0 \leq n \leq N$, that
$$\int_{\gamma_{n/N}} f(z)\,dz = \int_{\gamma_0} f(z)\,dz.$$

This is true for $n=0$. Assume $n \geq 1$ and the equality holds for $\gamma_{\frac{n-1}{N}}$; it is enough then to check that

$$\int_{\gamma_{n/N}} f(z)\,dz = \int_{\gamma_{\frac{n-1}{N}}} f(z)\,dz.$$

For $0 \leq m \leq N$, let

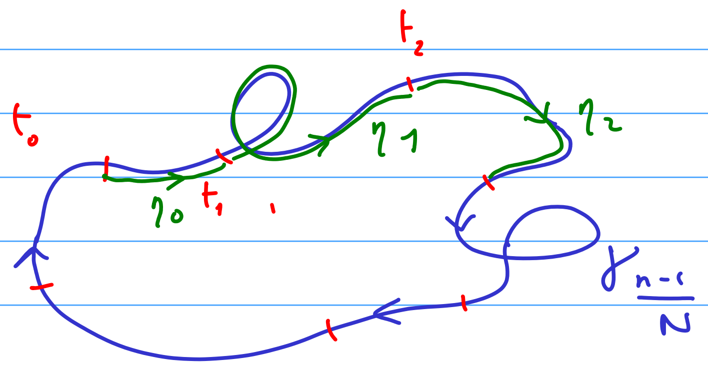
σ_m be the (oriented) segment



between $F(x_{m,n-1})$ and $F(x_{m,n})$; it is contained in U, since the whole small square is mapped to a disc contained in U. For $0 \le m < N$, let

$$\eta_m \quad , \quad \eta_m'$$

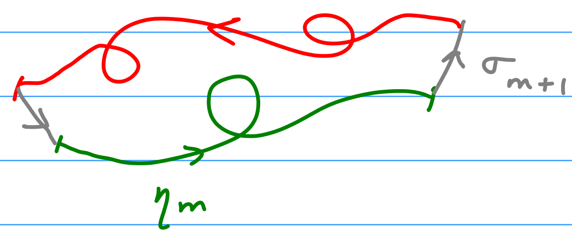
denote respectively the restriction of $\gamma_{\frac{n-1}{N}}$ and $\gamma_{n/N}$ to $[t_m, t_{m+1}]$.



Then by Cauchy's Theorem in $D_\delta(F(x_{n-1,m}))$:

$$\int_{\eta_m} f + \int_{\sigma_{m+1}} f - \int_{\eta_m'} f - \int_{\sigma_m} f = 0$$

so that



$$\int_{\gamma_{\frac{n-1}{N}}} f = \sum_{m=0}^{N-1} \int_{\eta_m} f$$

$$= \sum_{m=0}^{N-1} \int_{\eta_m'} f + \sum_{m=0}^{N-1} \left(\int_{\sigma_m} f - \int_{\sigma_{m+1}} f \right)$$

$$= \int_{\gamma_{n/N}} f + \int_{\sigma_0} f - \int_{\sigma_N} f$$

$$= \int_{\gamma_{n/N}} f$$

because $\sigma_0 = \sigma_N$ since $\begin{cases} \gamma_{\frac{n-1}{N}}(a) = \gamma_{\frac{n-1}{N}}(b) \\ \gamma_{n/N}(a) = \gamma_{n/N}(b). \end{cases}$

(the curves are closed).

$\square$

4. Simply-connected sets

Definition. [p° 96]

An open set $U \subset \mathbb{C}$ is simply connected if it is connected and if any two curves with the same endpoints are homotopic (with same endpoints).

Examples.

(1) $U = \mathbb{C}$

(2) U convex

(3) $U = \mathbb{C} -]-\infty, 0]$

$\Bigg\}$ are simply connected

(4) But $\mathbb{C} - \{0\}$ is not simply connected

From the homotopy theorem we get:

<u>Proposition</u>. Let $U \subset \mathbb{C}$ be simply connected and $f \in \mathcal{H}(U)$. For any smooth curve γ in U, the value $\int_\gamma f(z)dz$ <u>only depends</u> on the endpoints of γ. In particular

$$\int_\gamma f(z)dz = 0$$

if γ is a closed curve.

<u>Corollary</u>. Let $U \subset \mathbb{C}$ be simply connected and $f \in \mathcal{M}(U)$ meromorphic on U. Let γ be a smooth curve in U of the form

$$\gamma(t) = z_0 + r(t)\, e^{i\theta(t)}$$

for $0 \leq t \leq 2\pi$ with r of class C^1 s.t. $\begin{cases} r(t) > 0 \\ r(0) = r(2\pi) \end{cases}$ and θ of class C^1 with

$$\theta(0) = \theta(2\pi).$$

Assume that no pole of f is on γ.

Then $\displaystyle\int_\gamma f(z)\,dz = 2i\pi \sum_{\substack{\text{pole } z\\ \text{inside}\\ \gamma}} \operatorname{res}_z(f)$.

<u>Proof.</u> This follows from the argument in Chapter $\underline{\mathrm{V}}$, p. 14.

<u>Theorem.</u> Let $\emptyset \neq U \subset \mathbb{C}$ be simply connected. Then any $f \in \mathcal{H}(U)$ has a primitive, well-defined up to adding a constant.

<u>Proof.</u> Let $z_0 \in U$. Because U is open and connected (part of the definition) there is some smooth curve γ_z joining z_0 to z (cf. Exercise Sheet 3, exercise 1: the function $\varphi : U \to \{0,1\}$ mapping z to 1 if and only if such γ_z exists is continuous, so constant, so everywhere equal to 1 because z close to z_0 can be joined by a segment).

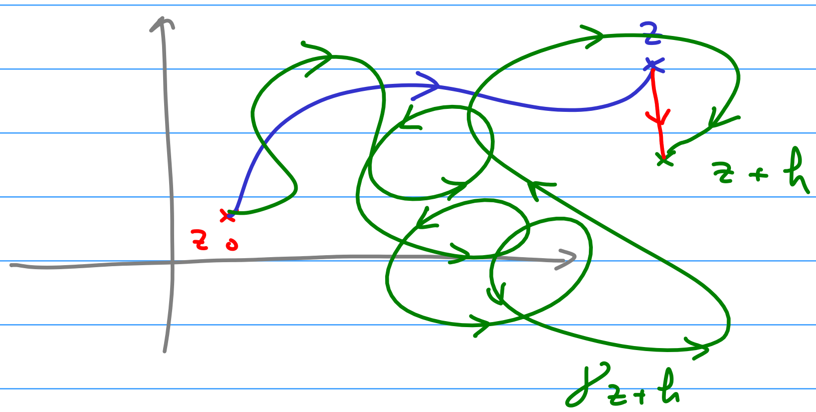
Define $\quad F(z) = \int_{\gamma_z} f(w)\,dw \quad , \quad z \in U.$

Then for $z \in U$ and δ such that $D_\delta(z)$ is contained in U, we get

$$F(z+h) - F(z) = \int_{[z,\, z+h]} f(w)\,dw$$

(because the curve γ_{z+h}, followed by the segment to z, followed by γ_z reversed is a closed curve) and then, as in Chapter III, p°. 5, we deduce that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$



$$0 = \int_{\gamma_z} f + \int_{[z,\, z+h]} f - \int_{\gamma_{z+h}} f$$

$$= F(z) - F(z+h) + \int_{[z,\, z+h]} f$$

$\square$

5. The complex logarithm

Definition. Let $U \subset \mathbb{C}$ be open. A **branch of the logarithm** $\log_U$ on U is a function in $\mathcal{H}(U)$ such that $\exp(\log_U(z)) = z$ for all $z \in U$.

Ex. Since $\exp(z) \neq 0$ for all $z \in \mathbb{C}$, this can only exist if $0 \notin U$.

If $U = \mathbb{C} - \{0\}$, however, although

$$\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\}$$

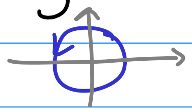
is surjective $(\forall z, \exists w, \exp(w) = z)$ there is no holomorphic choice of logarithm.

Indeed if there was $f \in \mathcal{H}(\mathbb{C} - \{0\})$ with $\exp(f(z)) = z$ for all z, we would get $f'(z)\exp(f(z)) = 1$ for all z, so

$$f'(z) = \frac{1}{z}, \quad z \in U.$$

But then we would need to have

$$0 = \int \frac{1}{z}\, dz = 2i\pi \ldots$$



<u>Proposition.</u> If $U \subset \mathbb{C}^*$ is simply connected then there exists a branch of the logarithm on U.

<u>Proof.</u> Let f be a primitive of the function $1/z$, which is holomorphic on U.

Let $\varphi(z) = \exp(f(z))$.

Then $\varphi'(z) = f'(z) \exp(f(z)) = \dfrac{\varphi(z)}{z}$

$$\varphi''(z) = \frac{z\varphi'(z) - \varphi(z)}{z^2} = 0$$

for all $z \in U$. This means that φ' is constant, so $\varphi(z) = \alpha z + \beta$ for some α, β in $\mathbb{C}$. From $\varphi'(z) = \dfrac{\varphi(z)}{z}$ we get

$$\alpha = \alpha + \frac{\beta}{z}$$

so $\beta = 0$. We have $\alpha \in \mathbb{C}^\times$; write $\alpha = e^\beta$ for some $\beta \in \mathbb{C}$, and then note that

$$\exp(f(z) - \beta) = \varphi(z) e^{-\beta} = z$$

for all $z \in U$.

$\square$

Definition. The principal branch of the logarithm is the unique branch of the logarithm on $\mathbb{C} -]-\infty, 0]$ such that $\log(1) = 0$.

This exists and is unique because if ℓ_1 is a branch of the logarithm on $U = \mathbb{C} -]\infty, 0]$ (which is simply connected) then there exists $k \in \mathbb{Z}$ such that $\ell_1(1) = 2ik\pi$, and then

$$\ell_2(z) = \ell_1(z) - 2ik\pi$$

is a branch of the logarithm with $\ell_2(1) = 0$. It is unique, because if ℓ_3 is another, then

$$k(z) = \ell_3(z) - \ell_2(z) \in 2i\pi\mathbb{Z}$$

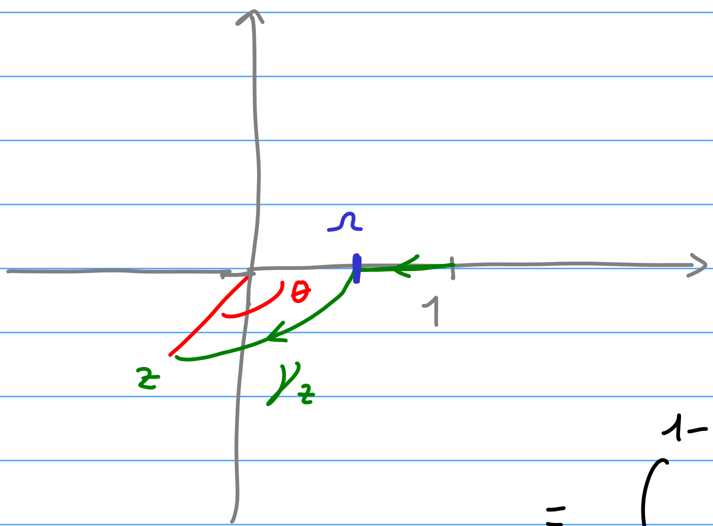
for all $z \in U$, and k is continuous, so must be constant, so equal to 0.

(p. 99)

<u>Proposition</u>. For $z = r e^{i\theta}$ with $r > 0$ and $-\pi < \theta < \pi$, then $\log(z) = \log(r) + i\,\theta$.

principal branch

<u>Proof</u>. Consider the curve γ_z below.



Then

$$\log(z) = \int_{\gamma_z} \frac{dw}{w}$$

(because this sends 1 to 0)

$$= \int_0^{1-r} \frac{-dx}{1-x} \qquad (r < 1)$$

segment
$z = 1 - x$
$(0 \leq x \leq 1-r)$

$$+ \int_0^{-\theta} \frac{-ir}{r e^{-it}} e^{-it}\, dt$$

arc
$z = r e^{-it}$, $0 \leq t \leq -\theta$

$$= \log(r) - i \int_0^{-\theta} dt = \log(r) + i\,\theta.$$

If $r > 1$, a similar computation gives the same result.

□

Remark. Let $U \subset \mathbb{C}^\times$ be simply connected and $\log_U : U \longrightarrow \mathbb{C}$ be a branch of logarithm.

For $\alpha \in \mathbb{C}$ and $z \in U$, we then define

$$z^\alpha = \exp(\alpha \log_U z).$$

Note that this depends on the choice of $\log_U$!

(If $\log_U$ is replaced by $\log_U + 2ik\pi$, then z^α becomes $\exp(\alpha(\log_U z + 2ik\pi)) = e^{2ik\pi\alpha} z^\alpha$)

In particular if $\alpha = \frac{1}{m}$ for $m \geqslant 1$ integer, we get $z^{1/m}$, which has the property that

$$\left(z^{1/m}\right)^m = \exp\left(m \cdot \frac{1}{m} \log_U(z)\right)$$

$$= \exp(\log_U(z)) = z.$$

(E.g. for $m = 2$, a "branch of the squareroot").

6. Winding numbers and residues

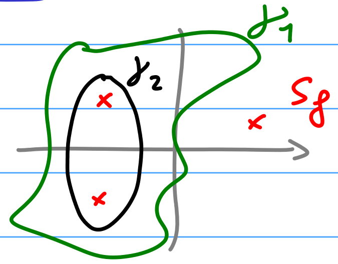
The goal here is to understand line integrals

$$\int_\gamma f(z)\,dz$$

when f is meromorphic and γ closed. A first generalization of the residue formula is:

<u>Proposition.</u> Let $U \subset \mathbb{C}$ be open. Let $f \in \mathcal{M}(U)$ and $V = U - S_f$, so that $f \in \mathcal{H}(V)$.

(1) Let γ_1 and γ_2 be <u>closed</u> curves $\boxed{\text{in } V} \subset U$ which are homotopic $\boxed{\text{in } V}$. Then



$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

(2) If γ_2 is a circle with counterclockwise orientation then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz = 2i\pi \sum_{\substack{z_0 \in \gamma_2 \\ z_0 \in S_f}} res_{z_0}(f).$$

<u>Proof.</u> (1) is a special case of the Homotopy Theorem, (2) follows then from the residue formula.

□

To handle more general curves, we have the following definition.

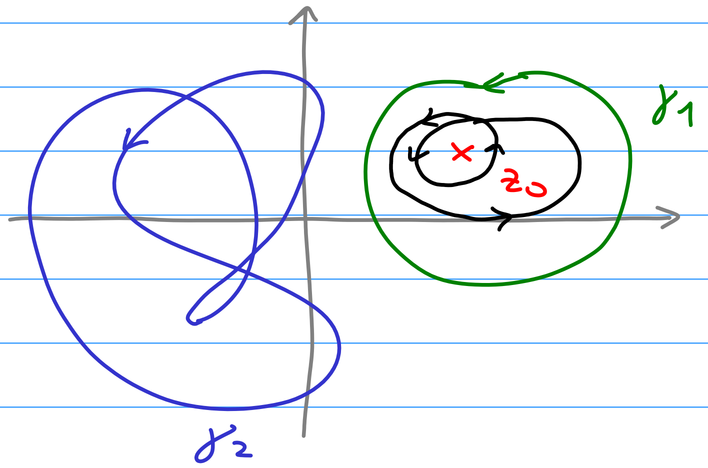
Definition. (Winding number; p° 347)

Let $z_0 \in \mathbb{C}$, γ closed curve in $\mathbb{C}$ st. $z_0 \notin \gamma$.

The winding number (or index) of γ around z_0 is

$$W_\gamma(z_0) = \frac{1}{2i\pi} \int_\gamma \frac{dz}{z - z_0}.$$

Example. (1) Intuitively, $W_\gamma(z_0)$ is "the number of times γ winds around z_0". So in the following picture, we should get:



$$W_{\gamma_1}(z_0) = 1 \quad ; \quad W_{\gamma_2}(z_0) = 0 \quad ;$$

$$W_{\gamma_3}(z_0) = 2.$$

This is indeed the case. But to begin with we compute $W_\gamma(z_0)$ if is a circle taken counterclockwise, $k \geq$ times, i.e.

$$\gamma(t) = z_0 + r e^{it} \qquad (r > 0)$$

for $0 \leq t \leq 2k\pi$. Then $\gamma'(t) = i r e^{it}$

$$W_\gamma(z_0) = \frac{1}{2i\pi} \, i \int_0^{2k\pi} \frac{r e^{it}}{z_0 + r e^{it} - z_0} \, dt$$

$$= \frac{1}{2\pi} \cdot 2k\pi = k.$$

(2) Now let $\gamma(t) = z_0 + e^{it}$, $0 \leq t \leq 2\pi$, but take the winding number at a different $z_1 \notin \gamma$.

We get

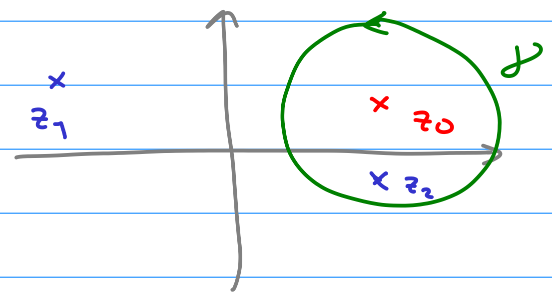
$$W_\gamma(z_1) = \begin{cases} 1 & \text{if } |z_1 - z_0| < r, \\ 0 & \text{otherwise} \end{cases}$$

which is consistent with the "winding" idea.

To see this, we can just apply the Residue Formula to the function

$$f(z) = \frac{1}{z - z_1} \in \mathcal{M}(\mathbb{C})$$

which has a unique pole at z_1 with residue 1;



in the

second case, we note that z_1 is <u>not</u> inside the disc bounded by γ.

[B.1.3]

<u>Proposition</u>. Let γ be a closed curve in $\mathbb{C}$, and $U \subset \mathbb{C}$ the open set $\mathbb{C} - (\text{image of } \gamma)$. The map $W_\gamma : U \longrightarrow \mathbb{C}$ takes values in $\mathbb{Z}$ and is continuous (so it is constant on any connected open subset $V \subset U$). Moreover $W_\gamma(z) = 0$ if $|z|$ is large enough.

<u>Proof</u>. Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a parameteri-zation of γ. Define $F : [a,b] \longrightarrow \mathbb{C}$ by

$$F(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t) - z} \, dt$$

for some fixed z not in γ, so that $W_\gamma(z) = \dfrac{F(b)}{2i\pi}$.

By the fundamental theorem of calculus, we see that F is continuous and differentiable on $]a,b[$ with $F'(x) = \dfrac{\gamma'(x)}{\gamma(x) - z}$. It follows that $\big((\gamma(x) - z)\exp(-F(x))\big)' = 0$ by the

Chain Rule (it is $\gamma'(x)\exp(-F(x))$

$+ (\gamma(x) - z) \cdot \left(-\frac{\gamma'(x)}{\gamma(x) - z}\right) \exp(-F(x))$)

hence $\gamma(x) - z = c \exp(F(x))$ for all x for some constant c. Then

$$c = \underbrace{c \exp(F(a))}_{=1} = \gamma(a) - z = \gamma(b) - z$$
$$= c \exp(F(b))$$

so that $\exp(F(b)) = 1$, hence $F(b) \in 2ik\pi\mathbb{Z}$.

So w_γ takes values in $\mathbb{Z}$. It is continuous (integral of continuous function), so it is locally constant.

Finally, if $M = \sup |\gamma(t)|$ then

$$|w_\gamma(z)| \leq \frac{1}{2\pi} \frac{\text{length}(\gamma)}{M - |z|}$$

for $|z| > M$, which goes to 0. Since $w_\gamma(z)$ must be zero once $|w_\gamma(z)| < 1$, we get the last result.

$\square$

Theorem. Let $U \subset \mathbb{C}$ be simply connected. Let $f \in \mathcal{M}(U)$ and $V = U - S_f$. Let γ be a closed curve in V. We have

$$\int_\gamma f(z)\,dz = 2i\pi \sum_{z_0 \in S_f} W_\gamma(z_0)\,\mathrm{res}_{z_0}(f).$$

Proof. For any $z_0 \in S_f$, let p_{z_0} be the principal part of f at z_0 (so

$$p_{z_0}(z) = \sum_{j=1}^{v(z_0)} \frac{a_j(z_0)}{(z-z_0)^j} \quad \text{for some } a_j(z_0)$$

in $\mathbb{C}$, where $v(z_0)$ is the order of the pole at z_0).

Assume first that S_f is finite. Then

$$\tilde{f} = f - \sum_{z_0 \in S_f} p_{z_0}$$

is holomorphic on U (because $p_{z_0} \in \mathcal{M}(\mathbb{C}-\{z_0\})$ and removing it "eliminates" the pole at z_0). We get $\int_\gamma \tilde{f}(z)\,dz = 0$ because U is simply connected, so

$$\int_\gamma f(z)\,dz = \sum_{z_0 \in S_f} \int_\gamma p_{z_0}(z)\,dz$$

Now we note that

$$\int_\gamma \frac{dz}{(z-z_0)^j} = 0 \quad \text{if} \quad j \neq 1$$

because $\frac{1}{(z-z_0)^j}$ has the primitive $-\frac{1}{j-1}\frac{1}{(z-z_0)^{j-1}}$,

so

$$\int_\gamma f(z)\,dz = \sum_{z_0 \in S_f} \int_\gamma \frac{a_1(z_0)}{z-z_0}\,dz$$

$$= \sum_{z_0 \in S_f} a_1(z_0) \cdot 2i\pi\, W_\gamma(z_0).$$

Now consider the general case where S_f might be infinite. Pick $R > 0$ so that $W_\gamma(z) = 0$ if $|z| \geq R$ and so that γ is homotopic to the constant curve $\gamma_2(t) = \gamma(a)$ in $U \cap D_R(0)$ (this R exists by the proposition and the fact that a homotopy exists in U, which "involves" only a bounded subset).

Then $S_f \cap D_R(0)$ is finite; let

$$\tilde{f} = f - \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} p_{z_0} \in \mathcal{H}(U \cap D_R(0)).$$

We have $\int_\gamma \tilde{f} = 0$ because of the homotopy in $U \cap D_R(0)$, so

$$\int_\gamma f = \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} \int_\gamma p_{z_0} .$$

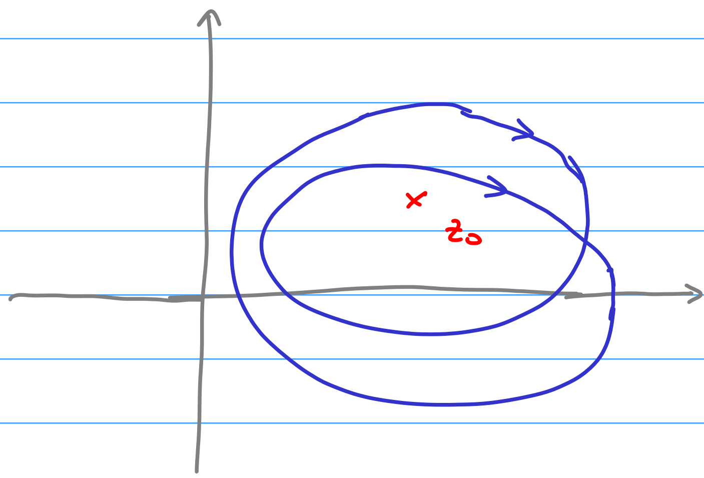
Arguing as in the first case, we obtain

$$\int_\gamma f = 2i\pi \sum_{\substack{z_0 \in S_f \\ |z_0| < R}} W_\gamma(z_0)\, \mathrm{res}_{z_0}(f)$$

$$= 2i\pi \sum_{z_0 \in S_f} W_\gamma(z_0)\, \mathrm{res}_{z_0}(f)$$

since $W_\gamma(z_0) = 0$ if $z_0 \in S_f$ with $|z_0| \geq R$.

$\square$

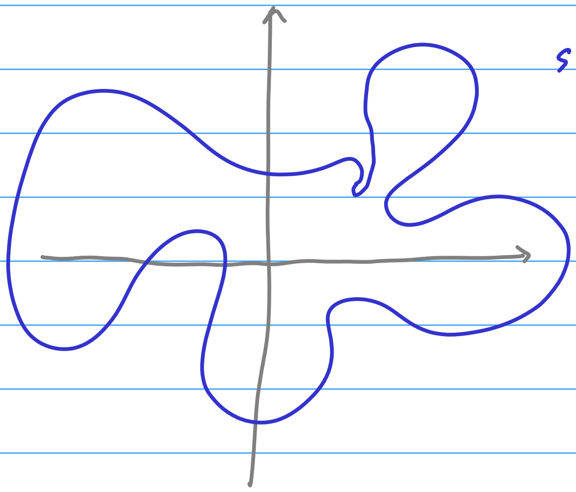


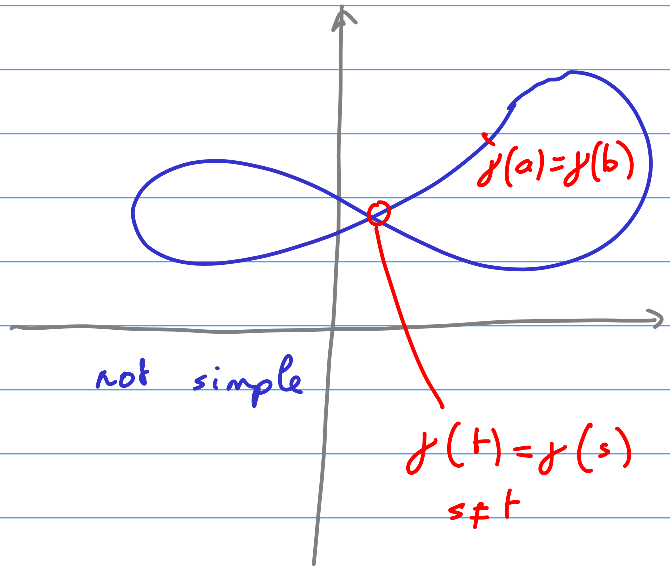
Example.

$$\int_\gamma f(z)\, dz = -4i\pi\, \mathrm{res}_{z_0}(f)$$

(because γ turns twice, clockwise, around z_0)

Remark. Winding numbers provide a way to determine when $z \in \mathbb{C}$ is "inside" a curve. Indeed, let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a simple closed curve, which means that $\gamma(t) \neq \gamma(s)$ unless $s=t$ or $\{s,t\} = \{a,b\}$.





Intuitively, one "sees" that $\mathbb{C} - \gamma$ has two parts, the "interior" and the "exterior". These can be characterized by the value of $W_\gamma(z)$: if z is in the "exterior", then $W_\gamma(z) = 0$ (in any case, this is true for $|z|$ large) and the interior by $|W_\gamma(z)| = 1$ (the sign depends on the orientation).

These facts are not easy to prove rigorously! See for instance Appendix B in the book of Stein and Shakarchi, especially Th. 2.2.