Chapter VIII

Conformal mapping

1 - Definition and examples

The question we consider next is to "compare" different open sets.

Definition - Let U, V be open sets in $\mathbb{C}$. An injective holomorphic map $f : U \longrightarrow V$ is called a conformal map from U to V. If f is bijective then we say that it is a conformal equivalence (or holomorphic isomorphism, or biholomorphism), and that U, V are conformally equivalent.

If $U = V$, a conformal equivalence is an automorphism.

Before giving examples, the following is important:

Proposition - (VIII.1.1) If $f : U \longrightarrow V$ is conformal, then $f'(z)$ is non-zero for all $z \in U$ (but the converse is not true e.g. $f(z) = e^{z}$)

If $f : U \longrightarrow V$ is a conformal equivalence, then the reciprocal bijection $f^{-1} : V \longrightarrow U$ is also. In particular "conformal equivalence" is an equivalence relation, denoted $\sim_c$:

$$U \sim_c U \qquad \text{for any } U$$

$$U \sim_c V \iff V \sim_c U$$

$$U \sim_c V \text{ and } V \sim_c W \implies U \sim_c W$$

Proof. We prove that if $z_0 \in U$ satisfies $f'(z_0) = 0$, then f is not injective. Let $k = \operatorname{ord}_{z_0}(f - f(z_0))$; we have $k \geq 2$ by assumption. If $k = +\infty$, then f is constant on a disc around z_0, so not injective. Assume k is finite; then

$$f(z) = f(z_0) + \underbrace{\frac{f^{(k)}(z_0)}{k!}}_{= \alpha \neq 0} (z - z_0)^k + g(z)(z - z_0)^{k+1}$$

where g is holomorphic on some $D_r(z_0) \subset U$.

Let $w \in \mathbb{C}$. Then

$$f(z) - f(z_0) - w = \left(\alpha (z - z_0)^k - w\right) + g(z)(z - z_0)^{k+1}.$$

We apply Rouché's Theorem as follows: let $C = \sup_{|z-z_0| = \frac{r}{2}} |g(z)|$, which exists because g is continuous. Pick $0 < s < \frac{r}{2}$, $s < 1$, and assume that $|w| < \frac{|\alpha| s^k}{2}$.

Then $$|\alpha(z-z_0)^k - w| \geq \frac{|\alpha| s^k}{2} \quad \text{if } |z-z_0| = s.$$

On the other hand

$$|(z-z_0)^{k+1} g(z)| \leq C s^{k+1}$$

so we can apply Rouché's Theorem as soon as $Cs < \frac{|\alpha|}{2}$ to conclude that the equation

$$f(z) = f(z_0) + w$$

has the same number of solutions with $|z-z_0| < s$ as the equation $\alpha(z-z_0)^k = w$. For $|w|$ small enough (in particular so that $|w| < \frac{|\alpha| s^k}{2}$) there are k solutions. If $w \neq 0$ then these are $\neq z_0$; then if z_1 is a solution $f'(z_1)$ will be non-zero (because z_0 is an isolated zero of f') so the solutions are distinct. So f is not injective.

From this the remainder of the proposition comes easily: if $f^{-1} : V \longrightarrow U$ is the reciprocal bijection of a conformal equivalence then we get

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}$$

$w = f(z)$
$w_0 = f(z_0)$

$$\Big\downarrow \; w \to w_0$$

$$\frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}$$

so $f^{-1} \in \mathcal{H}(V)$.

The last steps are elementary, e.g. if $U \sim_c V$ and $V \sim_c W$, with $U \xrightarrow{f} V$ and $V \xrightarrow{g} W$ conformal equivalences, then $g \circ f : U \longrightarrow W$ is a holomorphic bijection, so a conformal equivalence.

$\square$

Corollary. If $U \sim_c V$ by $f : U \longrightarrow V$, then the map $\mathcal{H}(V) \xrightarrow{\alpha} \mathcal{H}(U)$ defined by

$$U \xrightarrow{f} V \xrightarrow{\varphi} \mathbb{C}$$

$$\alpha(\varphi) = \varphi \circ f$$

is a linear isomorphism with inverse $\alpha^{-1}(\varphi) = \varphi \circ f^{-1}$.

It also satisfies $\alpha(\varphi_1 \varphi_2) = \alpha(\varphi_1)\alpha(\varphi_2)$.

Proof. Elementary from the proposition and direct computations.

$\square$

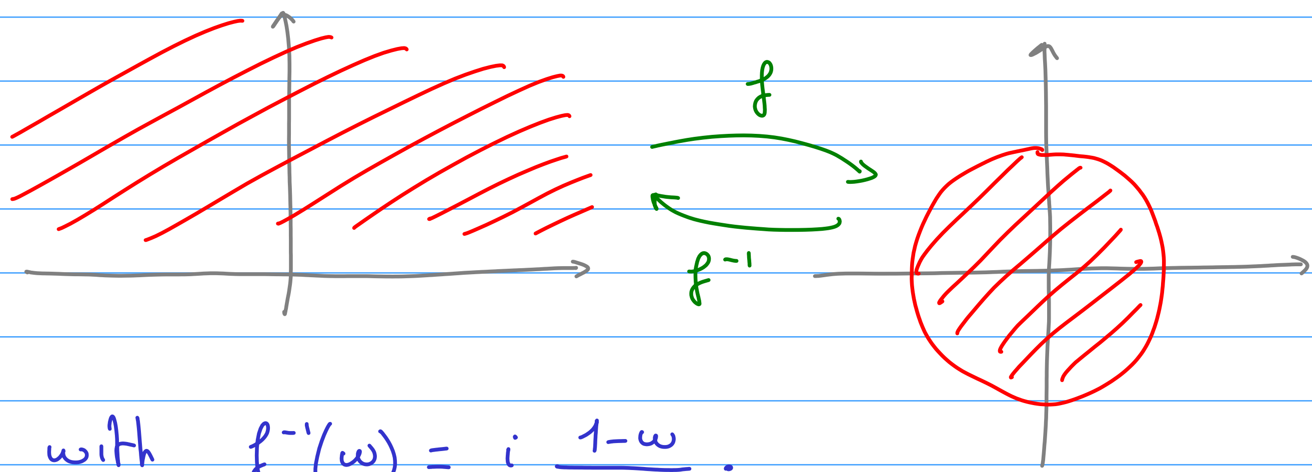
Examples. (1) Let $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$.

Fact: the holomorphic map

$$f(z) = \frac{i-z}{i+z}$$

for $z \in \mathbb{H}$ is a conformal equivalence

$$\mathbb{H} \longrightarrow D_1(0)$$



with $f^{-1}(w) = i\,\dfrac{1-w}{1+w}$.

(In particular, the property that a set is bounded is $\underline{\underline{\text{not}}}$ preserved by conformal equivalence.)

<u>Proof</u>. We have $f \in \mathcal{H}(\mathbb{H})$ and $|f(z)| < 1$ for $z \in \mathbb{H}$ (the distance to i is less than the distance to $-i$). Similarly, $g : D_1(0) \longrightarrow \mathbb{C}$ defined by $g(w) = i\,\frac{1-w}{1+w}$ is holomorphic, and one computes

$$|\operatorname{Im} g(w)| = \frac{1-|w|^2}{|1+w|^2} > 0,$$

so it defines a map $D_1(0) \longrightarrow \mathbb{H}$ which is holomorphic. One computes then

$$f(g(w)) = \frac{\cancel{i} - \cancel{i}\,\frac{1-w}{1+w}}{\cancel{i} + \cancel{i}\,\frac{1-w}{1+w}} = \frac{\frac{1+w-(1-w)}{\cancel{1+w}}}{\frac{1+w+1-w}{\cancel{1+w}}}$$

$$= \frac{2w}{2} = w$$

for $|w| < 1$, and similarly

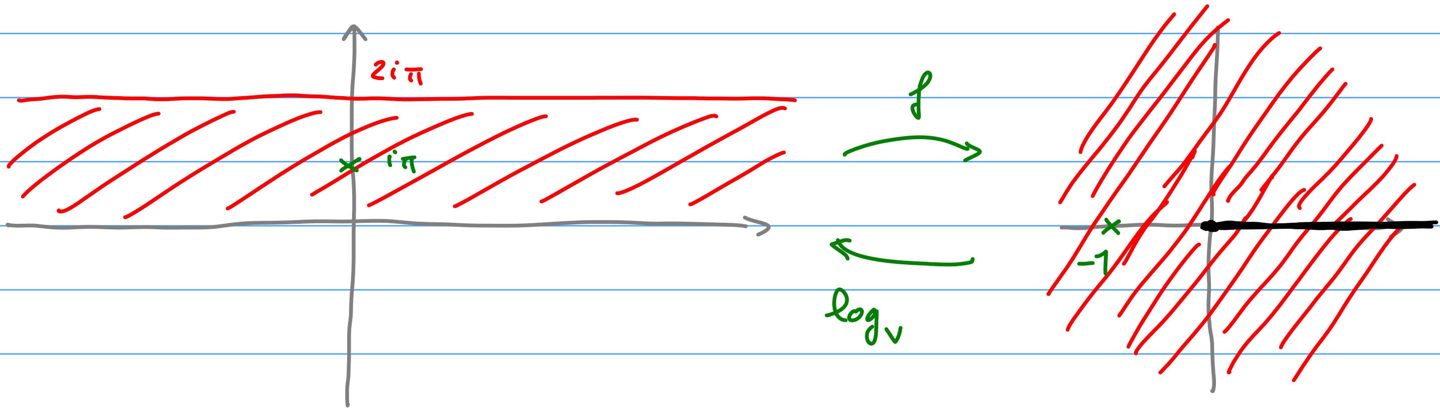
$$g(f(z)) = i\,\frac{1 - \frac{i-z}{i+z}}{1 + \frac{i-z}{i+z}} = i\,\frac{\frac{i+z-(i-z)}{\cancel{i+z}}}{\frac{i+z+i-z}{\cancel{i+z}}}$$

$$= i \cdot \frac{2z}{2i} = z$$

for $z \in \mathbb{H}$.

(2) Let $f(z) = e^z$ for $z \in U = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) \in]0, 2\pi[\}$.

Then f is a conformal map (since $e^z = e^w \iff z - w \in 2i\pi\mathbb{Z}$), and it gives a conformal equiva-
-lence $U \longrightarrow \mathbb{C} - [0, +\infty[= V$



(where $\log_V$ is normalized by $\log_V(-1) = i\pi$).

(3) Let $U = \mathbb{C}$ and $V = D_1(0)$. Then U and V are <u>not</u> conformally equivalent.

Indeed, if we have $f : U \longrightarrow V$ holomorphic, then since f is holomorphic and bounded, it is constant by Liouville's Theorem.

2 - Riemann's Theorem

We will prove: (VIII . 3.1)

<u>Theorem</u> (Riemann) Let $U \subset \mathbb{C}$ be a <u>non-empty</u>

simply connected open set. Then $U \underset{c}{\simeq} D_1(0)$.

More precisely, for any $z_0 \in U$, there exists a unique conformal equivalence $f : U \longrightarrow D_1(0)$ such that $f(z_0) = 0$ and $f'(z_0) \in]0, +\infty[\subset \mathbb{C}$.

Remark. This Theorem therefore classifies all simply-connected open sets $U \subset \mathbb{C}$ up to conformal equivalence: there are three cases:

$$\begin{cases} \emptyset \\ \mathbb{C} \\ D_1(0) \end{cases}.$$

From the example before, we know these are pairwise non-equivalent.

The strategy of the proof is the following:

(1) Step 0: uniqueness (this more or less means finding the automorphisms of $D_1(0)$, since having $f_1, f_2 : U \xrightarrow{\sim} D_1(0)$ implies that $f_2 \circ f_1^{-1}$ is an automorphism of $D_1(0)$).

(2) Step 1: if $U \neq \mathbb{C}$, there is a conformal

map $f : U \longrightarrow D_1(0)$ (so U is conformally equivalent to some open subset of $D_1(0)$)

(3) Step 2 : There exists a conformal map
$$f : U \longrightarrow D_1(0)$$
with $f(z_0) = 0$ and $|f'(z_0)|$ maximal.

(4) Step 3 : the f of Step 2 is surjective.

Among these steps, we will see that Step 2 is the most original and involved.

3 - Automorphisms and uniqueness

[VIII. 2.2]

Theorem - Let $f : D_1(0) \longrightarrow D_1(0)$ be an auto-morphism. Then there exist $\theta \in \mathbb{R}$ and $\alpha \in D_1(0)$ such that $f(z) = e^{i\theta} \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ for all z.
We then have $f(0) = e^{i\theta}\alpha$ and $f'(0) = e^{i\theta}(|\alpha|^2 - 1)$.
Conversely, all such maps are automorphisms.

Before giving the proof, we see how it gives

The uniqueness in Riemann's Theorem: if f_1 and f_2 are conformal equivalences $U \xrightarrow{f_1, f_2} D_1(0)$ then $g = f_2 \circ f_1^{-1}$ is of the form above for some $\theta \in \mathbb{R}$ and $\alpha \in D_1(0)$; further the assumption $f_1(z_0) = f_2(z_0) = 0$ means that $g(0) = 0$, so $\alpha = 0$, hence $g(z) = e^{i\theta} z$ for $z \in D_1(0)$; then

$$g'(0) = (f_1^{-1})'(0) \cdot f_2'(z_0) = \frac{f_2'(z_0)}{f_1'(z_0)} = e^{i\theta}$$

and the other assumption gives $e^{i\theta} > 0$ so $e^{i\theta} = 1$.

<u>Lemma</u> - (Schwarz's Lemma; III. 2. 1)

Let $f : D_1(0) \longrightarrow D_1(0)$ be holomorphic with $f(0) = 0$. Then

(a) We have $|f(z)| \leq |z|$ for all z;

(b) If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then there exists $\theta \in \mathbb{R}$ s.t. $f(z) = e^{i\theta} z$.

(c) $|f'(0)| \leq 1$ with equality if and only if there exists $\theta \in \mathbb{R}$ s.t. $f(z) = e^{i\theta} z$.

Proof. (a) Since $\mathrm{ord}_0 f \geq 1$, we can define a function $g \in \mathcal{H}(D_1(0))$ by $g(z) = \frac{f(z)}{z}$.

For $0 < r < 1$ and $|z| = r$, we get

$$|g(z)| = \frac{1}{r} |f(z)| \leq \frac{1}{r}$$

(since $f(z) \in D_1(0)$) so $|g(z)| \leq \frac{1}{r}$ for all $z \in \overline{D}_r(0)$ (by the Maximum Modulus Principle), hence $|f(z)| \leq \frac{|z|}{r}$ for $0 \leq |z| \leq r$.

Letting $r \to 1$ for $z \in D_1(0)$ fixed, this gives

$$|f(z)| \leq |z|.$$

(b) Note that (a) gives $\sup_{z \in D_1(0)} |g(z)| \leq 1$, and the assumption then gives $|g(z_0)| = \sup_{z \in D_1(0)} |g(z)|$ so $z_0 \in D_1(0)$ is a local maximum of g; by the Maximum Modulus Principle again, the function g must be constant. Thus there exists $c \in \mathbb{C}$ s.t.

$$f(z) = z\, g(z) = c\, z \quad \text{for all } c \in D_1(0).$$

Moreover $c = g(z_0)$ so $|c| = 1$.

(c) We have $g(0) = \lim_{z \to 0} \frac{f(z)}{z} = f'(0)$,

so first $|f'(0)| = |g(0)| \leq 1$, and second, if $|f'(0)| = 1$, then again 0 is a local maximum of g, and we conclude as in (b).

$\square$

<u>Proof of classification</u>. We first check that

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \bar{\alpha} z}$$

is an automorphism. First, $1 - \bar{\alpha} z \neq 0$ if $|z| < 1$ so $f \in \mathcal{H}(D_1(0))$. Next

$$f(z) = f(w) \iff \frac{\alpha - z}{1 - \bar{\alpha} z} = \frac{\alpha - w}{1 - \bar{\alpha} w}$$

$$\iff \cancel{\alpha} - |\alpha|^2 w - z + \cancel{\bar{\alpha} z w} = \cancel{\alpha} - |\alpha|^2 z - w + \cancel{\bar{\alpha} z w}$$

$$\iff (1 - |\alpha|^2) w = (1 - |\alpha|^2) z$$

$$\iff w = z$$

so $f : D_1(0) \longrightarrow \mathbb{C}$ is a conformal map.

In fact, $f \in \mathcal{H}(D_r(0))$ where $r = \frac{1}{|\alpha|} > 1$, and one sees that if $z = e^{it}$, then

$$|f(z)| = \left| \frac{\alpha - e^{it}}{1 - \bar{\alpha} e^{it}} \right| = \frac{|\alpha - e^{it}|}{|e^{it}(e^{-it} - \bar{\alpha})|}$$

$$= 1$$

so $|f(z)| < 1$ if $z \in D_1(0)$ by the Maximum Modulus Principle. Hence f gives a conformal map $D_1(0) \xrightarrow{f} D_1(0)$. And finally

$$f(z) = w \iff \frac{\alpha - z}{1 - \bar{\alpha} z} = e^{-i\theta} w$$

$$\iff \alpha - z = e^{-i\theta} w - \bar{\alpha} e^{-i\theta} z w$$

$$\iff z(\bar{\alpha} e^{-i\theta} w - 1) = -\alpha + e^{-i\theta} w$$

$$\iff z = e^{-i\theta} \frac{\alpha e^{i\theta} - w}{1 - \bar{\alpha} e^{-i\theta} w},$$

and the RHS is $g(w)$ for some other function of the same type. So f is surjective.

Now we come back to a general automorphism $f : D_1(0) \longrightarrow D_1(0)$. Let $\alpha = f^{-1}(0)$.

Consider $g(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$; then $f \circ g$ is again an automorphism with $\begin{cases} f \circ g(0) = f(\alpha) = 0, \\ (f \circ g)^{-1}(0) = g^{-1}(\alpha) = 0 \end{cases}$

By the Schwarz Lemma, we get

$$|f(g(z))| \leq |z| \quad \text{for} \quad z \in D_1(0)$$

and

$$|g^{-1}(f^{-1}(w))| \leq |w| \quad \text{for} \quad w \in D_1(0).$$

In particular for $w = f(g(z))$ we get

$$|z| \leq |f(g(z))|$$

so finally $|f(g(z))| = |z|$ for $z \in D_1(0)$.

Again from the Schwarz Lemma, this gives some $\theta \in \mathbb{R}$ s.t. $f(g(z)) = e^{i\theta} z$ for all z

so $f(z) = e^{i\theta} g^{-1}(z)$, which has the desired form according to the previous computation.

$\square$

<u>Remark.</u> If we use the conformal equivalence $\mathbb{H} \xrightarrow{f} D_1(0)$ given by $f(z) = \frac{i-z}{i+z}$, one can deduce from the previous description that any automorphism de $\mathbb{H}$ is given by

$$g(z) = \frac{az+b}{cz+d}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ such that $ad-bc>0$.

Moreover, for $h(z) = \frac{a'z+b'}{c'z+d'}$, $a'd'-b'c'>0$

one checks that $g \circ h(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ with

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}_{\text{ordinary matrix product}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

4_ <u>U is conformally equivalent to a subset of $D_1(0)$</u>

<u>Proposition</u>_ Let $U \subset \mathbb{C}$ with $U \neq \mathbb{C}$ be a ($\emptyset^*$) simply connected open set. There exists a conformal map $f : U \longrightarrow D_1(0)$ such that $0 \in f(U)$.

<u>Proof</u>_ There exists by assumption $\alpha \in \mathbb{C}$, $\alpha \notin U$. By replacing U by $\{z-\alpha \mid z \in U\}$, we can assume $\alpha = 0$ so $U \subset \mathbb{C} - \{0\}$. Since U is simply connected, there exists a branch of the logarithm $\log_U : U \to \mathbb{C}$.

Since $\exp(\log_U(z)) = z$, we see that $\log_U$ is injective, hence conformal.

Pick $z_0 \in U$. Then $\log_U(z_0) + 2i\pi \notin \log_U(U)$ (since $\log_U(z) = \log_U(z_0) + 2i\pi \Rightarrow z = z_0$ by taking the exponential), and in fact there exists $\delta > 0$ such that

$$D_\delta(\log_U(z_0) + 2i\pi) \cap \log_U(U) = \emptyset.$$

[indeed, otherwise we get a sequence $z_n \in U$ with

$$\log_U(z_n) \longrightarrow \log_U(z_0) + 2i\pi$$

$$\Rightarrow z_n \longrightarrow z_0 \underset{(\log_U \text{ continuous})}{\Rightarrow} \log_U(z_n) \to \log_U(z_0)]$$

Define

$$\tilde{f} : U \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{1}{\log_U(z) - (\log_U(z_0) + 2i\pi)}$$

which is a conformal map with image contained in $D_{1/\delta}(0)$.

Finally, let $f(z) = \frac{\delta}{4}\left(\tilde{f}(z) - \tilde{f}(z_0)\right)$.

Then $f : U \longrightarrow \mathbb{C}$ is conformal, we have $f(z_0) = 0$ and

$$|f(z)| \leq \frac{\delta}{4} \cdot \left(\frac{1}{\delta} + \frac{1}{\delta} \right) \leq \frac{1}{2}$$

for all $z \in U$.

$\square$

5. Setting up an extremal problem

Consider again $U \neq \mathbb{C}$, simply connected, $U \neq \emptyset$, $z_0 \in U$.

By Section 4, there is a conformal map

$$U \xrightarrow{f} D_1(0)$$

such that $f(z_0) = 0$.

Let $\mathcal{F}$ be the set of all functions f with these properties.

Lemma. The set of values $|f'(z_0)|$ for $f \in \mathcal{F}$ is bounded in $[0, +\infty[$.

Proof. Let $\delta > 0$ be such that $D_{2\delta}(z_0) \subset U$.

For $f \in \mathcal{F}$, we have

$$f'(z_0) = \frac{1}{2i\pi} \int_{C_\delta(z_0)} \frac{f(z)}{(z - z_0)^2} \, dz$$

so that

$$|f'(z_0)| \leq \frac{1}{2\pi} \cdot 2\pi\delta \cdot \frac{1}{\delta^2} = \frac{1}{\delta}$$

since $|f(z)| \leq 1$ for all $z \in U$.

$\square$

Now we define

$$r = \sup_{f \in \mathcal{F}} |f'(z_0)| .$$

The key property to finish the proof is:

<u>Proposition</u>. There exists $f \in \mathcal{F}$ such that $|f'(z_0)| = r$.

To see why this is <u>the</u> crucial step, we now prove:

<u>Proposition</u>. Let $f \in \mathcal{F}$ be such that $|f'(z_0)| = r$. Then f is a conformal equivalence $U \longrightarrow D_1(0)$.

<u>Proof</u>. We need to show that f is surjective. We will do it by showing that if $\alpha \in D_1(0)$ is <u>not</u> in the image of f, then we can construct

$g \in \mathcal{F}$ with $|g'(z_0)| > |f'(z_0)|$, which is a contradiction.

So assume α exists. Let $\varphi : D_1(0) \longrightarrow D_1(0)$ be the automorphism $\varphi(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$; we have

$$\varphi(0) = \alpha, \qquad \varphi(\alpha) = 0.$$

Then $\varphi \circ f : U \longrightarrow D_1(0)$ is a conformal map such that $0 \notin \varphi \circ f(U)$; because U is simply connected we can find $\tilde{f} : U \longrightarrow \mathbb{C}$ holomorphic such that $\tilde{f}(z)^2 = \varphi(f(z))$ for all $z \in U$

[take $\tilde{f} = \exp\left(\frac{1}{2} \tilde{g}(z)\right)$ where $\tilde{g}$ is a primitive of $\dfrac{(\varphi \circ f)'}{\varphi \circ f}$, so that $\exp \circ \tilde{g}(z) = \varphi \circ f(z)$].

Note $\tilde{f}$ is also injective: $\tilde{f}(z) = \tilde{f}(w)$ implies

$\varphi(f(z)) = \varphi(f(w))$, hence $z = w$.

Let now $\beta = \tilde{f}(z_0)$ and

$$\psi(z) = \frac{\beta - z}{1 - \bar{\beta} z},$$

another automorphism of $D_1(0)$ with $\psi(0) = \beta$,

$\psi(\beta) = 0$. Put finally

$$g = \psi \circ \tilde{f} : D_1(0) \longrightarrow D_1(0).$$

This function belongs to $\mathcal{F}$: it is injective (because $\tilde{f}$ is), holomorphic, and

$$g(z_0) = \psi(\tilde{f}(z_0)) = \psi(\beta) = 0.$$

We claim that $|g'(0)| > |f'(0)|$: this will give the contradiction.

To prove the claim, we write

$$f = \varphi^{-1} \circ (\psi^{-1} \circ g)^2$$

$$\left[\quad g = \psi \circ \tilde{f} \quad \Rightarrow \quad \psi^{-1} \circ g = \tilde{f} \atop \quad \Rightarrow \quad (\psi^{-1} \circ g)^2 = \tilde{f}^2 = \varphi \circ f \quad \right]$$

and express it in the form

$$f = \Phi \circ g$$

where $\Phi = \varphi^{-1} \circ sq \circ \psi^{-1}$, $sq(z) = z^2$.

Note that $\Phi : D_1(0) \longrightarrow D_1(0)$ is holomorphic; by the Schwarz Lemma, (c), we get $|\Phi'(0)| \leq 1$; if there was equality, Φ would be a rotation

$\left(\Phi(z) = e^{i\theta} z \right)$ but that is not possible because

sq would then be injective.

So $|\Phi'(0)| < 1$, hence

$$f'(z_0) = \Phi'(g(z_0))\, g'(z_0)$$
$$= \Phi'(0)\, g'(z_0)$$

implies that $|g'(z_0)| > |f'(z_0)|$, as claimed.

$\square$

6. Existence of the maximum

The proof of the key Proposition is based on the following fairly natural idea:

(1) by definition, there is a sequence $(f_n)_{n \geq 1}$ in $\mathcal{F}$ such that $|f_n'(z_0)| \longrightarrow r$

(2) if (f_n) converges, uniformly on compact sets, to a function $f : U \longrightarrow \mathbb{C}$, then $f \in \mathcal{F}$ and $|f'(z_0)| = r$

(3) <u>even</u> if (f_n) doesn't converge uniformly on compact sets, at least some <u>subsequence</u> (f_{n_k}) does.

We now establish these facts; since (1) is true by definition of sup, we start with (2).

Proposition. Let (f_n) be a sequence in $\mathcal{F}$ and suppose that $f_n(z) \longrightarrow f(z)$ for $z \in U$, uniformly on any compact sets $K \subset U$. Then either f is constant or $f \in \mathcal{F}$. Also $f'(z_0) = \lim f_n'(z_0)$.

Proof. By the Convergence Theorem (Chapter IV, p° 19 and 21), we know that $f \in \mathcal{H}(U)$, and that f'_n also converges uniformly on compact sets to f, so $f'(z_0) = \lim f'_n(z_0)$.

There remains to prove that $|f(z)| < 1$ for all $z \in U$ and that f is injective or constant. For the first, from $|f_n(z)| < 1$ we deduce that $|f(z)| \leq 1$; but if there was equality at some $z \in U$, then z would be a local maximum of $|f|$, which is impossible by the Maximum Modulus Principle. So

$|f(z)| < 1$ for all $z \in U$.

Finally, the useful Lemma below shows that f is either injective or constant.

$\square$

<u>Note</u> - If $f_n'(z_0)$ does not converge to 0, then f cannot be constant. This is the case if $|f_n'(z_0)|$ converges to r, because $r > 0$: given that there exists $f \in \mathcal{F}$, we have $r \geq |f'(z_0)|$ and $f'(z_0)$ is non-zero since f is injective (cf. Prop., p°1).

<u>Lemma</u> - (VIII. 3. 5)

Let $U \subset \mathbb{C}$ be connected and open. Let $f_n : U \to \mathbb{C}$ be <u>conformal</u> maps. If (f_n) converges to $f : U \to \mathbb{C}$ locally uniformly, then f is either injective or constant.

<u>Proof</u> - We suppose that f is not injective, and will deduce that f is constant. The assumption

means that there exist $z_1 \neq z_2$ in U such that

$$f(z_1) = f(z_2).$$

If f is not constant, we can find a disc $D_\delta(z_2)$ contained in U s.t. $f(z) \neq f(z_2)$ for $z \in C_{\delta/2}(z_2)$.

Hence we get

$$\frac{1}{2i\pi} \int_{C_{\delta/2}(z_2)} \frac{f'(z)}{f(z) - f(z_1)}\, dz \geqslant 1$$

counterclockwise

(the LHS is the nb. of zeros of $f - f(z_1)$ in $D_{\delta/2}(z_2)$).

But $f_n(z) \neq f_n(z_1)$ for all n and $z \in C_{\delta/2}(z_2)$

(f_n is injective and $z_1 \notin C_{\delta/2}(z_2)$) so

$$\frac{f_n'}{f_n - f_n(z_1)} \longrightarrow \frac{f'}{f - f(z_1)}$$

uniformly on $C_{\delta/2}(z_2)$ and therefore

$$\frac{1}{2i\pi} \int_{C_{\delta/2}(z_2)} \frac{f'(z)}{f(z) - f(z_1)}\, dz = \lim_{n\to\infty} \frac{1}{2i\pi} \int_{C_{\delta/2}(z_2)} \frac{f_n'(z)}{f_n(z) - f_n(z_1)}\, dz$$

which is impossible since for each n the integral on the RHS counts the roots of $f_n(z) = f_n(z_1)$ in $D_{\delta/2}(z_2)$, and there are none by injectivity. $\square$

We are left with only the last step: finding a convergent subsequence.

7. Montel's Theorem

In fact, a much more general result holds.

Theorem ("Montel's Theorem"; ~~VIII~~.3.3)

Let $U \subset \mathbb{C}$ be an open set. Let (f_n) be a sequence in $\mathcal{H}(U)$. Suppose that:

for any compact set $K \subset U$, there exists $M_K \geq 0$ such that $|f_n(z)| \leq M_K$ for all $n \geq 1$ and $z \in K$.

Then there exists a subsequence (f_{n_k}) which converges uniformly on compact subsets of U.

In application to Riemann's Theorem, we have a sequence (f_n) in $\mathcal{F}$, so $|f_n(z)| \leq 1$ for all z and all n (not only for compact sets), hence we certainly can apply this Theorem.

The proof of Montel's Theorem relies on another important convergence theorem.

Theorem (Ascoli - Arzela Theorem)

Let $X \subset \mathbb{R}^n$ be a compact (closed, bounded) set and $f_n : X \longrightarrow \mathbb{R}^m$ continuous functions on X. Suppose:

(i) $\exists\, x_0 \in X, \exists\, M, \forall n, |f_n(x_0)| \leq M$

(ii) (f_n) is equicontinuous:

$\forall \varepsilon > 0, \exists\, \delta > 0, \forall n \geq 1, \forall x \in X, \forall y \in X$

$$\|x - y\| < \delta \Rightarrow \|f_n(x) - f_n(y)\| < \varepsilon$$

Then there is a subsequence (f_{n_k}) which converges uniformly on X to some (continuous) $f: X \to \mathbb{R}^m$.

This results belongs properly to topology/functional analysis, so we will admit it to finish the proof of Montel's Theorem.

Lemma. $U \subset \mathbb{C}$ open; there exist compact subsets $X_k \subset U$ s.t. $X_k \subset X_{k+1}$ and such that if $K \subset U$ is compact, then $K \subset X_k$ for some integer k.

Let us also assume this and prove Montel's Th.

Step 1. For any fixed $K \subset U$ compact, there is a subsequence of (f_n) converging uniformly on K.

To prove this, we apply to $X = K \subset \mathbb{C} \simeq \mathbb{R}^2$ and $f_n : X \longrightarrow \mathbb{C} \simeq \mathbb{R}^2$ the Ascoli-Arzela Theorem. We note that the fact that (f_n) is uniformly bounded on K gives condition (i) of the Theorem. It remains to prove that (f_n) is equicontinuous. This is basically a consequence of a Lipschitz condition. Precisely, we first pick $r > 0$ so that $D_{3r}(z) \subset U$ for $z \in K$.

Then for z_1, z_2 in K with $|z_1 - z_2| < r$,

we write

$$f_n(z_1) - f_n(z_2) = \frac{1}{2i\pi} \int_{\gamma_{z_2}} \underbrace{f_n(w) \left(\frac{1}{w - z_1} - \frac{1}{w - z_2} \right)}_{= \frac{(z_1 - z_2) f_n(w)}{(w - z_1)(w - z_2)}} dw$$

(where $\gamma_{z_2} = C_{2r}(z_2)$ counterclockwise) by Cauchy's formula, so

$$|f_n(z_1) - f_n(z_2)| \leq \frac{1}{\cancel{2\pi}} \times \cancel{2\pi} \times 2r \times |z_1 - z_2|$$
$$\times M_K \times \frac{1}{2r^2}$$

$\forall w, \forall n, \ |f_n(w)| \leq M_K$

$|w - z_2| = 2r$

$|w - z_1| \geq |w - z_2| - |z_1 - z_2|$

$\geq 2r - r = r$

$$\leq \frac{M_K}{r} |z_1 - z_2|.$$

So for any $\varepsilon > 0$, we will get

$$|f_n(z_1) - f_n(z_2)| < \varepsilon$$

for all n, all $z \in K$, as soon as

$$|z_1 - z_2| < \min\left(r, \frac{\varepsilon r}{M_K}\right).$$

Step 2 – To get uniform convergence on all compact sets, we use a trick called a "diagonal argument".

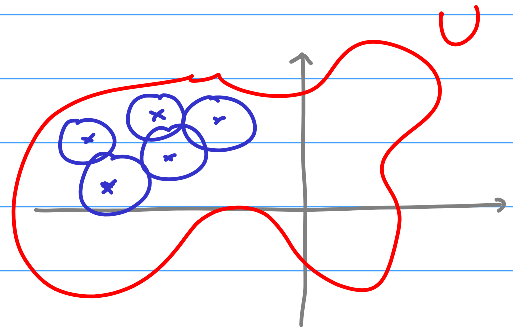
Let X_k be a sequence of compact subsets given by the last lemma.

By Step 1, there is a subsequence of (f_n) converging uniformly on X_1, say $(f_{n(1,m)})_{m\geq 1}$.

Then there is a subsequence of $(f_{n(1,m)})$ converging uniformly on X_2, hence on $X_1 \cup X_2$, say $(f_{n(2,m)})_{m\geq 1}$ (with $n(2,m) = n(1, \alpha_m)$ for some $\alpha_m > \alpha_{m-1} \dots$)

By induction, for any k, we get a sequence $(f_{n(k,m)})_{m\geq 1}$, with $n(k,m) = n(k-1, \alpha_m)$, such that $f_{n(k,m)}$ converges uniformly on $X_1, \dots, X_k$.

Finally: let $\beta_m = n(m,m)$; then (f_{β_m}) is a subsequence of all $(f_{n(k,m)})$, hence it converges uniformly on all X_k. Since any compact set is contained in one of the X_k, this concludes the proof. $\square$



Finally, we prove the lemma:

let $R = U \cap \mathbb{Q}^2$; there is a bijection

$$c : \mathbb{N} \longrightarrow R$$

(R is infinite and countable). For $z \in U$, let

$$\delta(z) = \operatorname{Sup}\{\delta > 0 \mid D_\delta(z) \subset U\}.$$

Let $$X_k = \bigcup_{1 \le j \le k} \overline{D}_{\delta(c(j))/2}(c(j)) \subset U.$$

This has the desired properties: indeed X_k is contained in X_{k+1}, X_k is compact; let $z \in U$; we can find $z_0 \in D_{\frac{\delta(z)}{4}}(z)$ and then $\delta(z_0) \geq \frac{\delta(z)}{2}$ (because, if w satisfies $|w - z_0| < \frac{\delta(z)}{2}$, then $|w - z| < \frac{\delta(z)}{2} + \frac{\delta(z)}{4}$)

so $|z - z_0| < \frac{\delta(z)}{4}$ gives $|z - z_0| < \frac{\delta(z_0)}{2}$

so $z \in D_{\frac{\delta(z_0)}{2}}(z_0)$; pick k s.t. $z_0 = c(k)$

then $z \in X_k$, so $U \subset \bigcup_{z_0 \in R} D_{\frac{\delta(z_0)}{2}}(z_0)$.

Finally, let $K \subset U$ be compact. From the open covering $K \subset \bigcup_{z_0 \in R} D_{\frac{\delta(z_0)}{2}}(z_0)$, we get a finite set $F \subset R$ such that

$$K \subset \bigcup_{z_0 \in F} D_{\frac{\delta(z_0)}{2}}(z_0)$$

and then $K \subset X_k$ as soon as

$$F \subset \{ c(j) \mid j \leq k \}.$$

$\square$