Exercise sheet 1

Exercise worth bonus points: Exercise 4

1. Determine the real and imaginary parts and write in polar form the following complex numbers:

   (a) $\dfrac{\sqrt{21}-\sqrt{7}+i(\sqrt{21}+\sqrt{7})}{2+2i}$,

   (b) $\left(\dfrac{\sqrt{2}+\sqrt{2}i}{\sqrt{2}-\sqrt{2}i}\right)^{2022}$,

   (c) $(1+i)^{2n}+(1-i)^{2n}$ for $n \in \mathbf{Z}_{\geqslant 0}$.

2. Let $a, b, c \in \mathbf{C}$. Find the solutions $z \in \mathbf{C}$ for the equation

   $$az + b\bar{z} + c = 0.$$

   When does it have exactly one solution?

3. For each of the following subsets of $\mathbf{C} = \mathbf{R}^2$, indicate whether they are (1) open, (2) closed, (3) compact, (4) connected:

   (a) $[0,1] \times \{0,1\}$,

   (b) $\{1/n \mid n \geqslant 1\}$,

   (c) $\{(1+1/n)^n \mid n \geqslant 1\} \cup \{e\}$,

   (d) $\{z \in \mathbf{C} \mid 2 < |z| \leqslant 3\}$.

4. **(Worth bonus points)** Let $\mathbf{H} = \{z \in \mathbf{C} \mid \operatorname{Im}(z) > 0\}$, and $\overline{\mathbf{H}} = \{z \in \mathbf{C} \mid \operatorname{Im}(z) < 0\}$.

   (a) Explain why $\mathbf{H}$ is open in $\mathbf{C}$.

   (b) For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c and d in $\mathbf{R}$ and $(c,d) \neq (0,0)$, show that the function $f\colon \mathbf{H} \to \mathbf{C}$ such that

   $$f(z) = \frac{az+b}{cz+d}$$

   is well-defined.

(c) Compute $\mathrm{Im}(f(z))$ and conclude that exactly one of the following conditions holds:

- f is constant;
- f maps $\mathbf{H}$ to $\mathbf{H}$;
- f maps $\mathbf{H}$ to $\overline{\mathbf{H}}$.

Find a simple necessary and sufficient condition on the matrix A for each of these cases to hold.

(d) Show that f is holomorphic on $\mathbf{H}$ and compute its derivative.

(e) Show that if f is not constant, then f is either a bijection from $\mathbf{H}$ to $\mathbf{H}$ or a bijection from $\mathbf{H}$ to $\overline{\mathbf{H}}$; compute then the inverse bijection of f. What do you observe?