Exercise sheet 10

Exercise worth bonus points: Exercise 5

1. Sketch the following open sets and show that they are simply connected:
   (a) $U_1 = \{x + iy \in \mathbf{C} \mid \text{ if } x = 0, \text{ then } y > 0\}$
   (b) $U_2 = \{x + iy \in \mathbf{C} \mid x > 0\}$
   (c) $U_4 = \{x + iy \in \mathbf{C} \mid 0 < y < x^2 + 1\}$.

2. Sketch the following open sets and show that they are not simply connected:
   (a) $U_5 = \{z \in \mathbf{C} \mid |z| > 1\}$
   (b) $U_6 = \{re^{i\theta} \in \mathbf{C} \mid 1/2 < r < 2\}$.

3. An open set $D \subset \mathbb{C}$ is said to be *star shaped* when there exists a point $z_0 \in D$, such that for every $z \in D$ the straight line segment between z and z_0 is contained in D.
   (a) Prove that a star shaped open set is connected.
   (b) Prove that a star shaped open set is simply connected.
   (c) Give an example of a simply connected open set that is not star shaped. (It is enough to make a picture and to explain why it works.)

4. Show that the Taylor expansion of the principal branch of the logarithm at $z_0 = 1$ is
$$\log(z) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(z-1)^n}{n}.$$

5. Let U be a non empty simply-connected open set in $\mathbf{C}$ and $f \in \mathcal{H}(U)$. We assume that $f(z) \neq 0$ if $z \in U$. Fix $z_0 \in U$. Recall that for any $z \in U$, there exists a smooth curve γ_z joining z_0 to z (Exercise 1 of Exercise Sheet 3).
   (a) Show that the function defined by
$$g(z) = \int_{\gamma_z} \frac{f'(w)}{f(w)} dw$$
   is holomorphic on U with $g'(z) = f'(z)/f(z)$ for all $z \in U$.

(b) Compute the derivative of $\exp(g(z))/f(z)$.

(c) Deduce that there exists a function $\widetilde{g} \in \mathcal{H}(U)$ such that $\exp(\widetilde{g}) = f$. Is it unique?

(d) Let $n \geqslant 1$. Show that there exists a function $h_n \in \mathcal{U}$ such that $h_n(z)^n = f(z)$ for all $z \in U$.