Exercise sheet 10

Exercise worth bonus points: Exercise 5

1. Sketch the following open sets and show that they are simply connected:

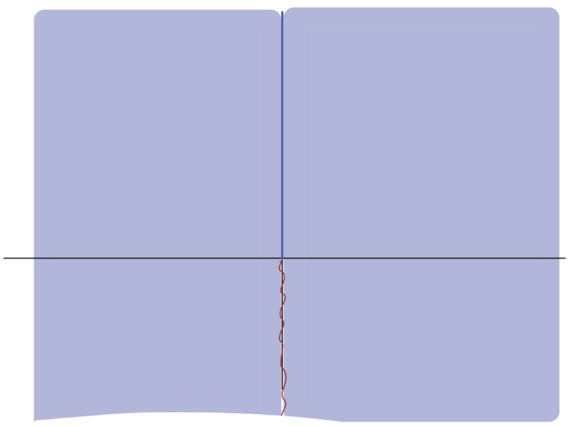
   (a) $U_1 = \{x + iy \in \mathbf{C} \mid \text{ if } x = 0, \text{ then } y > 0\}$

   (b) $U_2 = \{x + iy \in \mathbf{C} \mid x > 0\}$

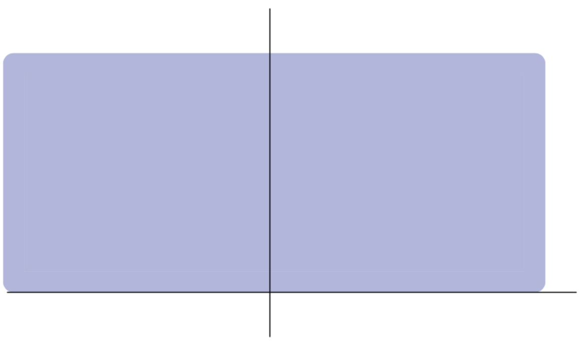
   (c) $U_4 = \{x + iy \in \mathbf{C} \mid 0 < y < x^2 + 1\}$.

   Solution:

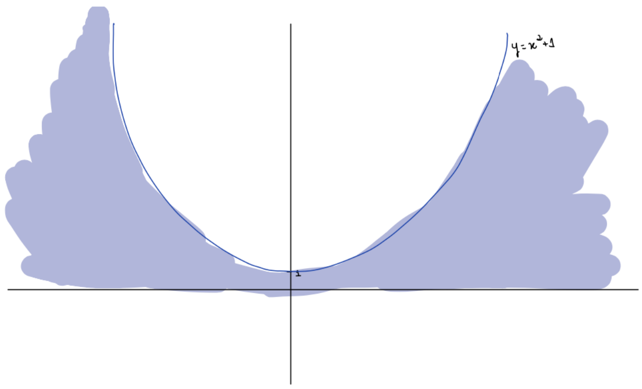
a)



b)



c)



(a) Observe that U_1 is a rotation of $\pi/2$ of the example seen in class $\mathbf{C} \setminus [0, +\infty)$ so one can adapt the proof seen in class to this example.

(b) U_2 is convex, so it must be simply connected.

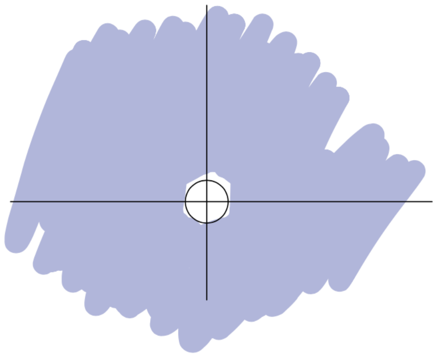
2. Sketch the following open sets and show that they are not simply connected:

   (a) $U_5 = \{z \in \mathbf{C} \mid |z| > 1\}$

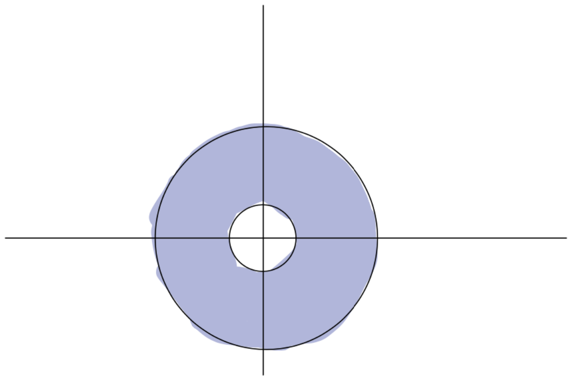
   (b) $U_6 = \{re^{i\theta} \in \mathbf{C} \mid 1/2 < r < 2\}$.

   Solution:

2. a)



b)



(a) Let $\gamma : [0, 2\pi] \to U_5$, $\gamma(t) = 2e^{it}$ and observe that

$$\int_\gamma \frac{1}{z} dz = 2\pi i \neq 0.$$

(b) Here the same holds for $\gamma : [0, 2\pi] \to U_6$, $\gamma(t) = e^{it}$. That is,

$$\int_\gamma \frac{1}{z} dz = 2\pi i \neq 0.$$

3. An open set $D \subset \mathbb{C}$ is said to be *star shaped* when there exists a point $z_0 \in D$, such that for every $z \in D$ the straight line segment between z and z_0 is contained in D.

   (a) Prove that a star shaped open set is connected.

   (b) Prove that a star shaped open set is simply connected.

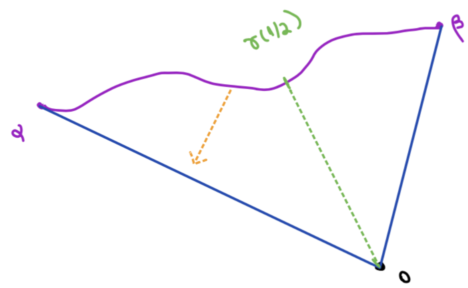
   (c) Give an example of a simply connected open set that is not star shaped. (It is enough to make a picture and to explain why it works.)

Bitte wenden.

Solutions:

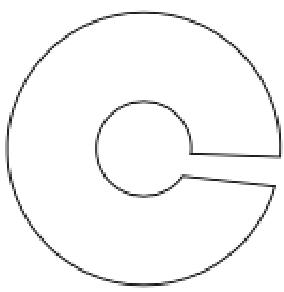
(a) We can assume, without loss of generality, that $z_0 = 0$. Thus, for every $z \in D$ we know that $tz \in D$, for $t \in [0,1]$.Suppose, by contradiction, that D is not connected, so we can write $D = U \cup V$, U, V disjoint open and non-empty sets. Let $0 \in U$ and $z_0 \in V$. Since U and V are open, we must have $tz \in U$ for $0 \leqslant t_0 < 1$ and $tz \in V$ for $t_1 < t \leqslant 1$, which gives us a contradiction, since the whole line must be contained in D and we cannot split it into two different open sets.

(b) Let $\gamma : [a,b] \to D$ be a curve connecting α and β. We observe that γ is homotopic to the curve that connects α to zero and then zero to β. We construct the homotopy by pushing the first half of the curve to the segment α to 0 and the second half to the segment 0 to β as pictured in the image.

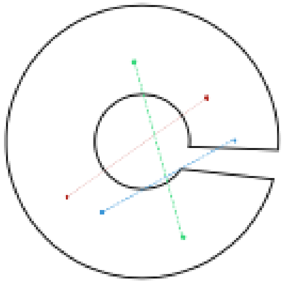


Since all curves with end points α and β are homotopic to the line that connected α to 0 and then 0 to β, we can conclude that any two such curves are homotopic.

(c) Let $U = \{re^{i\theta}$ for $1 < r < 2, 0 \leqslant 0 < 2\pi - \frac{1}{100}\}$.



Observe that U is simply connected. Given any point in $z \in U$, we can consider $z \cdot e^{(\pi+\varepsilon)i}$ for a $|\varepsilon| < \frac{1}{100}$ and observe that these points cannot be connected by a line inside U, as illustrated in the image. Thus, U can't be star shaped.



4. Show that the Taylor expansion of the principal branch of the logarithm at $z_0 = 1$ is

$$\log(z) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(z-1)^n}{n}.$$

Solution:

Observe that for $|z-1| < 1$ the sum on the right hand side is holomorphic. We know that the derivative is given by

$$\left(\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(z-1)^n}{n} \right)' = \sum_{n=1}^{+\infty} (-1)^{n+1} (z-1)^n = \frac{1}{z}.$$

So the derivaties coincide and thus

$$\log(z) + c = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(z-1)^n}{n}.$$

Since $\log(1) = 0$ and $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(1-1)^n}{n} = 0$ we conclude that $c = 0$ and the functions are equal.

5. Let U be a non empty simply-connected open set in $\mathbf{C}$ and $f \in \mathcal{H}(U)$. We assume that $f(z) \neq 0$ if $z \in U$. Fix $z_0 \in U$. Recall that for any $z \in U$, there exists a smooth curve γ_z joining z_0 to z (Exercise 1 of Exercise Sheet 3).

   (a) Show that the function defined by

$$g(z) = \frac{1}{2i\pi} \int_{\gamma_z} \frac{f'(w)}{f(w)} dw$$

   is holomorphic on U with $g'(z) = f'(z)/f(z)$ for all $z \in U$.

Bitte wenden.

(b) Compute the derivative of $\exp(g(z))/f(z)$.

(c) Deduce that there exists a function $\widetilde{g} \in \mathcal{H}(U)$ such that $\exp(\widetilde{g}) = f$. Is it unique?

(d) Let $n \geqslant 1$. Show that there exists a function $h_n \in \mathcal{U}$ such that $h_n(z)^n = f(z)$ for all $z \in U$.

Solution:

(a) First observe that the function is well-defined, since U is simply-connected and the integral doesn't depend on the chosen path. Observe that

$$g(z) - g(z+h) = \int_\eta \frac{f'(w)}{f(w)} dw,$$

where we can suppose that η is the straigh line that connects z to $z+h$ - which always exists for h small enough. We use continuity of $\frac{f'}{f}$ and write

$$\frac{f'(w)}{f(w)} = \frac{f'(z)}{f(z)} + \varphi(w),$$

for $\varphi(w) \to 0$ as $w \to z$. Thus

$$\lim_{h \to 0} \left(\frac{1}{h} \frac{f'(z)}{f(z)} \int_\eta dz + \frac{1}{h} \int_\eta \varphi(w) dw \right) = \frac{f'(z)}{f(z)}.$$

(b)

$$\begin{aligned} \left(\frac{\exp(g(z))}{f(z)} \right)' &= \frac{g'(z)\exp(g(z))f(z) - f'(z)\exp(g(z))}{f(z)^2} \\ &= \frac{\frac{f'(z)}{f(z)}\exp(g(z))f(z) - f'(z)\exp(g(z))}{f(z)^2} = 0. \end{aligned}$$

(c) Observe by the previous item that

$$\frac{\exp(g(z))}{f(z)} = c,$$

for c a non zero constant. Thus we can take c_0 such that $e^{c_0} = c$ and we conclude that $\tilde{g}(z) = g(z) - c_0$ satisfies $e^{\tilde{g}} = g$. Observe that we can sum to c_0 any integer multiple of $2\pi i$ and the result still works, so $\tilde{g}$ is not unique.

(d) Here we can take $h_n(z) = \exp(\frac{1}{n}\tilde{g}(z))$. From the previous items we have

$$(h_n(z))^n = \left(\exp\left(\frac{1}{n}\tilde{g}(z) \right) \right)^n \exp(\tilde{g}(z)) = f(z).$$