Exercise sheet 11

Exercise worth bonus points: Exercise 2

1. Let U be an open set in $\mathbf{C}$ and $z_0 \in U$, $r > 0$ such that $\overline{D}_r(z_0) \subset U$. Let $f \in \mathcal{H}(U)$ be such that $f(z) \neq 0$ for $z \in C_r(z_0)$. Show that for any $\varphi \in \mathcal{H}(U)$, we have

$$\frac{1}{2i\pi}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\varphi(z)dz = \sum_{\substack{z_0 \in D_r(z_0) \\ f(z_0)=0}} \mathrm{ord}_{z_0}(f)\varphi(z_0),$$

where the sum is over the zeros of f in the disc $D_r(z_0)$, and the circle is taken counterclockwise.

2. Let U be a connected open subset in $\mathbf{C}$ and $f\colon U \to \mathbf{C}$ a non-constant holomorphic function. Let $z_0 \in U$ and let k be the order of the zero of the function $g(z) = f(z) - f(z_0)$ at z_0.

   (a) Explain why $1 \leqslant k < +\infty$, and why there exists a simply connected neighborhood V of z_0, contained in U, such that $f(z) \neq f(z_0)$ if $z \in V$ and $z \neq z_0$.

   (b) Show that there exists a function $\varphi \in \mathcal{H}(V)$ such that $\varphi(z_0) = 0$ and

   $$f(z) = f(z_0) + \varphi(z)^k$$

   for all $z \in V$. (Hint: start by writing $f(z) = f(z_0) + (z - z_0)^k g(z)$ for some holomorphic function g not vanishing on V.)

   (c) Deduce that if $k \geqslant 2$, then f is *not* injective on V. (Hint: use the Open Image Theorem.)

3. Let $0 \leqslant s_1 < r_1 < r_2 < s_2$ be real numbers, and let U be the set

$$U = \{z \in \mathbf{C} \mid s_1 < |z| < s_2\},$$

and

$$V = \{z \in \mathbf{C} \mid r_1 < |z| < r_2\} \subset U.$$

We denote by γ_1, γ_2 the circles of radius r_1 and r_2, respectively, centered at 0, with counterclockwise orientation. Let $f \in \mathcal{H}(U)$.

(a) Show that the function g_1 defined by

$$g_1(z) = \frac{1}{2i\pi} \int_{\gamma_1} \frac{f(w)}{w - z} dw$$

is defined and holomorphic for $|z| > r_1$, and that the function g_2 defined by

$$g_2(z) = \frac{1}{2i\pi} \int_{\gamma_2} \frac{f(w)}{w - z} dw$$

is defined and holomorphic for $|z| < r_2$.

(b) Let γ be the closed curve obtained by going along the circle γ_2, starting at r_2, then taking the segment from r_2 to r_1, then going along the circle γ_1 with reversed (clockwise) orientation, and going along the segment from r_1 to r_2. Sketch γ.

Let z_0 be such that $r_1 < |z_0| < r_2$, and let σ be a circle of radius $\delta > 0$ small enough so that σ is contained in V. Explain why the closed curves γ and σ are homotopic in U (without giving a full proof, but sketching some steps of the homotopy).

(c) Show that $f(z) = g_2(z) - g_1(z)$ for $r_1 < |z| < r_2$.

(d) Show that there exist complex numbers a_n for $n \in \mathbf{Z}$ such that the series

$$\sum_{n \geqslant 1} a_n z^n \text{ and } \sum_{n \geqslant 1} a_{-n} z^{-n}$$

are both absolutely convergent for $r_1 < |z| < r_2$, and satisfy

$$f(z) = \sum_{n \in \mathbf{Z}} a_n z^n = \sum_{n \leqslant -1} a_n z^n + \sum_{n \geqslant 0} a_n z^n$$

for $r_1 < |z| < r_2$. (This is called the *Laurent expansion* of f around 0.)

(Hint: expand g_2 in power series, and expand $g_1(z^{-1})$ using geometric series.)