# Exercise sheet 11

**Exercise worth bonus points: Exercise 2**

1. Let $U$ be an open set in $\mathbf{C}$ and $z_0 \in U$, $r > 0$ such that $\overline{D}_r(z_0) \subset U$. Let $f \in \mathcal{H}(U)$ be such that $f(z) \neq 0$ for $z \in C_r(z_0)$. Show that for any $\varphi \in \mathcal{H}(U)$, we have

$$\frac{1}{2i\pi}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\varphi(z)dz = \sum_{\substack{w \in D_r(z_0) \\ f(w)=0}} \mathrm{ord}_w(f)\varphi(w),$$

where the sum is over the zeros of $f$ in the disc $D_r(z_0)$, and the circle is taken counterclockwise.

*Solution:*

Observe that if $w$ is a zero of order $n$ of $f$ we can write

$$f(z) = (z-w)^n g(z),$$

in a neighborhood of $w$ where $g$ is non vanishing. So,

$$\frac{f'(z)}{f(z)}\varphi(z) = \frac{n\varphi(z)}{z-w} + \frac{g'(z)}{g(z)}.$$

If $\varphi(w) = 0$ then $w$ is not a pole of $\frac{f'(z)}{f(z)}\varphi(z)$ and $\mathrm{ord}_w(f)\varphi(w) = 0$. Otherwise, $w$ is a pole of order 1 with residue $\mathrm{ord}_w(f)\varphi(w)$. So we can conclude, using the Residue Theorem, that

$$\frac{1}{2i\pi}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\varphi(z)dz = \sum_{\substack{w \in D_r(z_0) \\ f(w)=0}} \mathrm{ord}_w(f)\varphi(w).$$

2. Let $U$ be a connected open subset in $\mathbf{C}$ and $f\colon U \to \mathbf{C}$ a non-constant holomorphic function. Let $z_0 \in U$ and let $k$ be the order of the zero of the function $g(z) = f(z) - f(z_0)$ at $z_0$.

   (a) Explain why $1 \leqslant k < +\infty$, and why there exists a simply connected neighborhood $V$ of $z_0$, contained in $U$, such that $f(z) \neq f(z_0)$ if $z \in V$ and $z \neq z_0$.

(b) Show that there exists a function $\varphi \in \mathcal{H}(V)$ such that $\varphi(z_0) = 0$ and

$$f(z) = f(z_0) + \varphi(z)^k$$

for all $z \in V$. (Hint: start by writing $f(z) = f(z_0) + (z - z_0)^k g(z)$ for some holomorphic function $g$ not vanishing on $V$.)

(c) Deduce that if $k \geqslant 2$, then $f$ is *not* injective on $V$. (Hint: use the Open Image Theorem.)

*Solution:*

(a) Since $f$ is non-constant and $U$ is connected we know that there can't exist a sequence of points $z_n \to z_0$ such that $f(z_n) = f(z_0)$. Thus, there exists a neighborhood of $z_0$ $V$ such that $f(z) \neq f(z_0)$, $\forall z \in V \backslash \{z_0\}$.

(b) Expanding in Taylor series we know that we can write

$$f(z) = f(z_0) + (z - z_0)^k g(z),$$

for a $k \geqslant 1$ and $g$ a non-vanishing function in $V$. We can suppose without loss of generality that $V = B_r(z_0)$ for a $r > 0$, so $V$ is simply connected. Since $g$ is a non-vanishing function in a simply connected domain there exists a holormophic function $h$ such that

$$g(z) = e^{h(z)}.$$

We take $\varphi(z) = e^{\frac{h(z)}{k}}(z - z_0)$.

(c) Observe that $\varphi(z_0) = 0$. As a consequence of the Open Image Theorem, there exists a $\varepsilon > 0$ such that $B_{2\varepsilon}(0) \subset \varphi(V)$. So, there exists $w, \tilde{w} \in V$ such that

$$\varphi(w) = \varepsilon$$
$$\varphi(\tilde{w}) = \varepsilon e^{2\pi i/k}.$$

Since $k > 1$ we have $\varepsilon \neq \varepsilon e^{2\pi i/k}$ and

$$\varphi(w)^k = \varepsilon^k = \varepsilon^k e^{2\pi i} = \varphi(\tilde{w})^k.$$

3. Let $0 \leqslant s_1 < r_1 < r_2 < s_2$ be real numbers, and let $U$ be the set

$$U = \{z \in \mathbf{C} \mid s_1 < |z| < s_2\},$$

and

$$V = \{z \in \mathbf{C} \mid r_1 < |z| < r_2\} \subset U.$$

We denote by $\gamma_1, \gamma_2$ the circles of radius $r_1$ and $r_2$, respectively, centered at 0, with counterclockwise orientation. Let $f \in \mathcal{H}(U)$.

(a) Show that the function $g_1$ defined by

$$g_1(z) = \frac{1}{2i\pi} \int_{\gamma_1} \frac{f(w)}{w-z} dw$$

is defined and holomorphic for $|z| > r_1$, and that the function $g_2$ defined by

$$g_2(z) = \frac{1}{2i\pi} \int_{\gamma_2} \frac{f(w)}{w-z} dw$$

is defined and holomorphic for $|z| < r_2$.

(b) Let $\gamma$ be the closed curve obtained by going along the circle $\gamma_2$, starting at $r_2$, then taking the segment from $r_2$ to $r_1$, then going along the circle $\gamma_1$ with reversed (clockwise) orientation, and going along the segment from $r_1$ to $r_2$. Sketch $\gamma$.

Let $z_0$ be such that $r_1 < |z_0| < r_2$, and let $\sigma$ be a circle of radius $\delta > 0$ small enough so that $\sigma$ is contained in $V$. Explain why the closed curves $\gamma$ and $\sigma$ are homotopic in $U$ (without giving a full proof, but sketching some steps of the homotopy).

(c) Show that $f(z) = g_2(z) - g_1(z)$ for $r_1 < |z| < r_2$.

(d) Show that there exist complex numbers $a_n$ for $n \in \mathbf{Z}$ such that the series

$$\sum_{n \geqslant 1} a_n z^n \text{ and } \sum_{n \geqslant 1} a_{-n} z^{-n}$$

are both absolutely convergent for $r_1 < |z| < r_2$, and satisfy

$$f(z) = \sum_{n \in \mathbf{Z}} a_n z^n = \sum_{n \leqslant -1} a_n z^n + \sum_{n \geqslant 0} a_n z^n$$

for $r_1 < |z| < r_2$. (This is called the *Laurent expansion* of $f$ around 0.)

(Hint: expand $g_2$ in power series, and expand $g_1(z^{-1})$ using geometric series.)

*Solution:*

(a) We first observe that $g_1$ is continuous: if $z_k \to z$, $z_k, z \in \{z : |z| > r_1\}$ then

$$\begin{aligned} \lim_{k\to\infty} g_1(z_k) &= \lim_{k\to\infty} \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z_k} dw \\ &= \frac{1}{2\pi i} \int_{\gamma_1} \lim_{k\to\infty} \frac{f(w)}{w-z_k} dw = g_1(z), \end{aligned}$$

since $f(w)/(w-z)$ is continuous in $\{z : |z| > r_1\}$. We use that this function is also holomorphic in this domain, so given any triangle $\Delta$ in $\{z : |z| > r_1\}$:

$$\int_\Delta g_1(z)dz = \frac{1}{2\pi i}\int_{\gamma_1} f(w)\int_\Delta \frac{1}{w-z}dzdw = 0,$$

so it follows that $g_1$ is holomorphic by Morera's Theorem.

The proof that $g_2$ is holomorphic follows similarly.

(b)

(c) Using the previous result and Cauchy's Theorem we get

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{C_\delta(z)} \frac{f(w)}{w-z}dw = f(z).$$

To conclude just observe that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw = g_2(z) - g_1(z).$$

(d) For $|z| < r_2$ we can expand $g_2$ in power series:

$$g_2(z) = \sum_{n\geqslant 0} a_n z^n.$$

For $g_1$ we can write:

$$\begin{aligned} g_1(z) &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}dw = -\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{z}\cdot\frac{1}{1-\frac{w}{z}}dw \\ &= -\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{z}\sum_{n=0}^{+\infty}\left(\frac{w}{z}\right)^n dw \\ &= -\sum_{k=-1}^{-\infty}\left(\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w^{k+1}}dw\right)z^k = \sum_{k=-1}^{-\infty} a_k z^k. \end{aligned}$$