Exercise sheet 12

Exercise worth bonus points: Exercise 5

1. Let $f\colon U \to V$ be a conformal equivalence. Show that U is simply connected if and only if V is simply connected.

2. Show that the function $f(z) = (1+z)/(1-z)$ gives a conformal equivalence $U \to V$, where

$$U = \{z = x + iy \mid y > 0 \text{ and } x^2 + y^2 < 1\},$$
$$V = \{z = x + iy \mid x > 0 \text{ and } y > 0\}.$$

3. Let $\alpha \in]0, 2[$ be a real number.

   (a) Show that the function $f(z) = z^\alpha = \exp(\alpha \log(z))$, where log is the principal branch of the logarithm, is a conformal map from the upper half-plane $\mathbf{H}$ to $\mathbf{C}$.

   (b) Determine the image $f(\mathbf{H})$.

4. We define $f(z) = -\frac{1}{2}(z + 1/z)$, which is a meromorphic function on $\mathbf{C}$.

   (a) Show that for a given $w \in \mathbf{C}$, the equation $f(z) = w$ has two complex solutions with multiplicity, and determine when they are distinct.

   (b) Show that if $w \neq \pm 1$, then the equation $f(z) = w$ has two distict roots in $\mathbf{C}$.

   (c) Deduce that f defines a conformal map from

$$U = \{z = x + iy \mid y > 0 \text{ and } x^2 + y^2 < 1\}$$

   to $V = \mathbf{H}$.

5. Recall from Exercise 4 of Exercise Sheet 1 that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with real coefficients and determinant $ad - bc = 1$, then

$$f_A(z) = \frac{az + b}{cz + d}$$

   is a holomorphic bijection from $\mathbf{H}$ to $\mathbf{H}$, so it is an automorphism of $\mathbf{H}$.

(a) Show that $f_A(i) = i$ if and only if A is of the form

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

for some $\theta \in \mathbf{R}$.

(b) Let $z_0 \in \mathbf{H}$. Find a matrix B such that $f_B(i) = z_0$.

(c) Let f be an automorphism of $\mathbf{H}$. Show that there exists A such that $f \circ f_A(i) = i$.

(d) Deduce that there exists a matrix C such that $f = f_C$.