Exercise sheet 12

Exercise worth bonus points: Exercise 5

1. Let $f\colon U \to V$ be a conformal equivalence. Show that U is simply connected if and only if V is simply connected.

   Solution:

   Observe that it is enough to show that if U is simply connected then V is also simply connected, and the other direction follows because f^{-1} is also a conformal equivalence.

   Let $\gamma_1, \gamma_2 : [a,b] \to V$ be curves with same endpoints. Observe that $f^{-1} \circ \gamma_1$ and $f^{-1} \circ \gamma_2$ are curves in U with same endpoints. Since U is simply connected, there exists a homotopy H between them:

$$H : [a,b] \times [0,1] \to U$$
$$H(a,t) = H(a,0),\ H(b,t) = H(b,0)$$
$$H(x,0) = f^{-1} \circ \gamma_1(x)\ H(x,1) = f^{-1} \circ \gamma_2(x).$$

   We define a homotopy between γ_1 and γ_2 as follows:

$$\tilde{H} : [a,b] \times [0,1] \to V$$
$$\tilde{H}(x,t) = f(H(x,t)).$$

   Observe that $\tilde{H}$ is in fact a homotopy:

$$\tilde{H}(a,t) = f(H(a,t)) = f(f^{-1} \circ \gamma_1(a)) = \gamma_1(a)$$
$$\tilde{H}(b,t) = f(H(b,t)) = f(f^{-1} \circ \gamma_1(b)) = \gamma_1(b)$$
$$\tilde{H}(x,0) = f(f^{-1} \circ \gamma_1(x)) = \gamma_1(x)$$
$$\tilde{H}(x,1) = f(f^{-1} \circ \gamma_2(x)) = \gamma_2(x).$$

2. Show that the function $f(z) = (1+z)/(1-z)$ gives a conformal equivalence $U \to V$, where

$$U = \{z = x + iy \mid y > 0 \text{ and } x^2 + y^2 < 1\},$$
$$V = \{z = x + iy \mid x > 0 \text{ and } y > 0\}.$$

Solution:

It is clear that f is holomorphic in U. If we write $z = x + iy$ we have

$$f(z) = \frac{1-(x^2+y^2)}{(1-x)^2+y^2} + i\frac{2y}{(1-x)^2+y^2},$$

so we see that f maps U in the first quadrant V.

Moreover, the inverse map is given by $g(w) = \frac{w-1}{w+1}$. It holds that $|w-1| > |w+1|$ for all $w \in V$, since the distance of any point in V to $+1$ is smaller than the distance to -1. To conclude we just need to observe that the imaginary part of $g(w)$ is positive. It follows from

$$\operatorname{Im}\left(\frac{w-1}{w+1}\right) = \frac{2\operatorname{Im}(w)}{|w+1|^2}.$$

3. Let $\alpha \in]0, 2[$ be a real number.

   (a) Show that the function $f(z) = z^\alpha = \exp(\alpha \log(z))$, where log is the principal branch of the logarithm, is a conformal map from the upper half-plane $\mathbf{H}$ to $\mathbf{C}$.

   (b) Determine the image $f(\mathbf{H})$.

Solution:

(a) It is clear that f is holomorphic. We must show that it is injective. For $z \in \mathbf{H}$ we can write $z = re^{i\theta}$, for $r > 0$ and $0 < \theta < \pi$ and using the principal branch of the logarithm we get

$$f(z) = r^\alpha e^{i\theta\alpha}.$$

Let $z' = r'e^{i\theta'}$ and suppose that $f(z) = f(z')$. We get

$$r'^\alpha e^{i\theta'\alpha} = r^\alpha e^{i\theta\alpha},$$

so

$$r = r'$$
$$\alpha\theta = \alpha\theta' + 2\pi k,$$

$k \in \mathbf{Z}$. Since $0 < \alpha\theta, \alpha\theta' < 2\pi$ we get that $k = 0$ and $\theta = \theta'$ and $z = z'$.

(b) Observe that f takes the upper half plane to a sector $S = \{w \in \mathbf{C} : 0 < \arg(w) < \alpha\pi\}$.

4. We define $f(z) = -\frac{1}{2}(z + 1/z)$, which is a meromorphic function on $\mathbf{C}$.

   (a) Show that for a given $w \in \mathbf{C}$, the equation $f(z) = w$ has two complex solutions with multiplicity, and determine when they are distinct.

   (b) Show that if $w \neq \pm 1$, then the equation $f(z) = w$ has two distict roots in $\mathbf{C}$.

   (c) Deduce that f defines a conformal map from

   $$U = \{z = x + iy \mid y > 0 \text{ and } x^2 + y^2 < 1\}$$

   to $V = \mathbf{H}$.

Solution:

(a) Observe that we can rewrite the equality $f(z) = w$ as $z^2 + 2wz + 1 = 0$ and we know that the polynomial of degree 2 has two solutions with multiplicity. We have one solution with multiplicity two if an only if we can rewrite $z^2 + 2wz + 1 = (z - z')^2$, and comparing the coefficients we conclude that this only happens when $w = \pm 1$.

(b) Follows from the item above.

(c) Observe that f maps U to V:

$$\operatorname{Im}\left(-\frac{1}{2}\left(z + \frac{1}{z}\right)\right) = -\frac{1}{2}\left(\operatorname{Im}(z) - \frac{\operatorname{Im}(z)}{|z|^2}\right) > 0,$$

since $|z|^2 < 1$ and $\operatorname{Im}(z) > 0$. If $w \in \mathbf{H}$ then $f(z) = w$ has two distict solutions. Suppose one solution is in U. Then the other solution is given by $1/z$, which is not in U since $|\frac{1}{z}| > 1$. Thus, we conclude that f is injective from U to V.

5. Recall from Exercise 4 of Exercise Sheet 1 that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with real coefficients and determinant $ad - bc = 1$, then

   $$f_A(z) = \frac{az + b}{cz + d}$$

   is a holomorphic bijection from $\mathbf{H}$ to $\mathbf{H}$, so it is an automorphism of $\mathbf{H}$.

   (a) Show that $f_A(i) = i$ if and only if A is of the form

   $$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

   for some $\theta \in \mathbf{R}$.

(b) Let $z_0 \in \mathbf{H}$. Find a matrix B such that $f_B(i) = z_0$.

(c) Let f be an automorphism of $\mathbf{H}$. Show that there exists A such that $f \circ f_A(i) = i$.

(d) Deduce that there exists a matrix C such that $f = f_C$.

Solution:

(a) If we write $f_A(i) = i$ and compare the coefficients we get

$$\begin{aligned} a &= d \\ b &= -c. \end{aligned}$$

Since $\det(A) = 1$ we have $a^2 + b^2 = 1$ so we can write $a = \cos(\theta)$ and $b = \sin(\theta)$.

(b) We can do as in the previous item: writing $z_0 = x_0 + iy_0$ we get

$$\begin{aligned} b &= dx_0 - cy_0 \\ a &= cx_0 + dy_0 \\ ad - bc &= 1, \end{aligned}$$

so we have three equations and four variables, and thus there exists solutions a, b, c, d satisfying what we want.

(c) If f is an automorphism of $\mathbf{H}$ it there exists $z_0 \in \mathbf{H}$ such that $f(z_0) = i$. Let A be as in the previous item and observe that

$$f \circ f_A(i) = f(z_0) = i.$$

(d) Consider the map $f : \mathbf{H} \to \{z \in \mathbf{C} : |z| < 1\}$

$$F(z) = i\frac{z - i}{z + i}.$$

From Riemman Mapping Theorem we know that this is the unique conformal equivalence between $\mathbf{H}$ and the disk such that $F(i) = 0$ and $F'(i) > 0$. Observe that $g = (e^{i\beta}F) \circ f \circ f_A$ is a conformal equivalence between $\mathbf{H}$ and the disk. It holds that $g(i) = 0$ and we can choose β so that $g'(i) > 0$. Using the uniqueness of F we can write f as a Moebius Transform.