Exercise sheet 1

Exercise worth bonus points: Exercise 4

1. Determine the real and imaginary parts and write in polar form the following complex numbers:

   (a) $\dfrac{\sqrt{21}-\sqrt{7}+i(\sqrt{21}+\sqrt{7})}{2+2i}$,

   (b) $\left(\dfrac{\sqrt{2}+\sqrt{2}i}{\sqrt{2}-\sqrt{2}i}\right)^{2022}$,

   (c) $(1+i)^{2n}+(1-i)^{2n}$ for $n \in \mathbf{Z}_{\geqslant 0}$.

   Solution:

   (a) We set
   $$z = \frac{\sqrt{21}-\sqrt{7}+i(\sqrt{21}+\sqrt{7})}{2+2i}$$
   Observe that we can write
   $$z = \frac{\sqrt{21}}{2} + i\frac{\sqrt{7}}{2},$$
   so we conclude that $|z| = \sqrt{7}$ and $\arg(z) = \frac{\pi}{6}$. Thus,
   $$z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{7}}{2} = \sqrt{7}e^{i\pi/6}.$$

   (b) Observe that
   $$z = \frac{\sqrt{2}+\sqrt{2}i}{\sqrt{2}-\sqrt{2}i} = \frac{w}{\bar{w}},$$
   so $|z| = 1$ and we can write
   $$z = e^{i2\cdot 2021 \arg(\sqrt{2}+\sqrt{2}i)} = e^{i2022\pi/2} = -1.$$

(c) We first write each componnent of the sum in polar coordinates, that is:

$$1 \pm i = \sqrt{2}e^{\pm i\frac{\pi}{4}}.$$

This implies that

$$\begin{aligned}(1+i)^{2n} + (1-i)^{2n} &= (\sqrt{2}e^{i\frac{\pi}{4}})^{2n} + (\sqrt{2}e^{-i\frac{\pi}{4}})^{2n} \\ &= 2^n(e^{i\pi\frac{n}{2}} + e^{-i\pi\frac{n}{2}}) = 2^{n+1}\cos\left(\frac{\pi n}{2}\right),\end{aligned}$$

and we conclude that

$$(1+i)^{2n} + (1-i)^{2n} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2^{n+1}\cos\left(\frac{\pi n}{2}\right) & \text{if } n \text{ is even.}\end{cases}$$

2. Let $a, b, c \in \mathbf{C}$. Find the solutions $z \in \mathbf{C}$ for the equation

$$az + b\bar{z} + c = 0.$$

When does it have exactly one solution?

Solution:

We rewrite the equation with $a = a_1 + ia_2$, $b = b_1 + ib_2$, $-c = c_1 + ic_2$ and $z = z_1 + iz_2$ so it becomes

$$(a_1 + b_1)z_1 - (a_2 - b_2)z_2 + i((a_2 + b_2)z_1 + (a_1 - b_1)z_2) = c_1 + ic_2.$$

This is equivalent to the equations system

$$\begin{pmatrix} a_1 + b_1 & -(a_2 - b_2) \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

We know that the system has exactly one solution when

$$\det \begin{pmatrix} a_1 + b_1 & -(a_2 - b_2) \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix} = a_1^2 - b_1^2 + a_2^2 - b_2^2 \neq 0$$

which is equivalent to $|a| \neq |b|$.

3. For each of the following subsets of $\mathbf{C} = \mathbf{R}^2$, indicate whether they are (1) open, (2) closed, (3) compact, (4) connected:

   (a) $[0, 1] \times \{0, 1\}$,

   (b) $\{1/n \mid n \geqslant 1\}$,

(c) $\{(1+1/n)^n \mid n \geqslant 1\} \cup \{e\}$,

(d) $\{z \in \mathbf{C} \mid 2 < |z| \leqslant 3\}$.

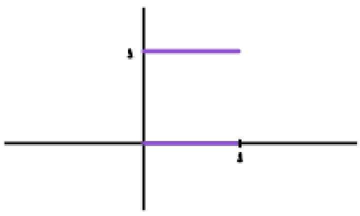
Solution:

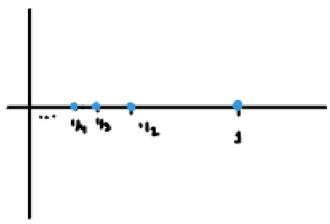
(a) Observe that the set is (2)closed and (3)compact. It is neither open nor connected.

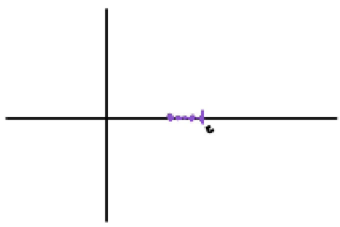
(b) Observe that the set is not opened, since for every point $z_k = 1/k$, every ball $B_r(z_k)$ with $r < \frac{1}{3k^2}$ is not contained in the set. It is also not closed, (consequently not compact) since the 0 does not belong to the set. And it is not connected.

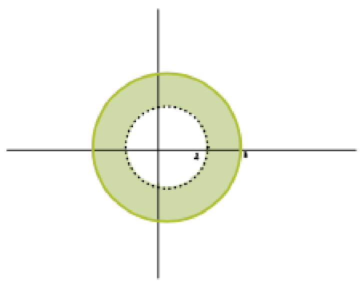
(c) This example is similar to the previous one, but in this case we the limit of the
sequence is added to the set. Thus, the set is closed and compact, but it is not connected nor open.

(d) Observe that the set is not opened because of the border $|z| = 3$ and it is not closed (nor compact) because it do not contain the border $|z| = 2$. The set is connected.









4. **(Worth bonus points)** Let $\mathbf{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$.

(a) Explain why $\mathbf{H}$ is open in $\mathbf{C}$.

(b) For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c and d in $\mathbf{R}$ and $(c, d) \neq (0, 0)$, show that the function $f\colon \mathbf{H} \to \mathbf{C}$ such that

$$f(z) = \frac{az + b}{cz + d}$$

is well-defined.

(c) Compute $\mathrm{Im}(f(z))$ and conclude that exactly one of the following conditions holds:

- f is constant;
- f maps $\mathbf{H}$ to $\mathbf{H}$;
- f maps $\mathbf{H}$ to $\bar{\mathbf{H}}$.

Find a simple necessary and sufficient condition on the matrix A for each of these cases to hold.

(d) Show that f is holomorphic on $\mathbf{H}$ and compute its derivative.

(e) Show that if f is not constant, then f is either a bijection from $\mathbf{H}$ to $\mathbf{H}$ or a bijection from $\mathbf{H}$ to $\mathbf{H}$; compute its inverse. What do you observe?

Solution:

(a) Let $z \in \mathbf{H}$. Since $\mathrm{Im}(z) > 0$ we can take ε sufficiently small such that $\mathrm{Im}(z) > \varepsilon$. Take the ball centered in z with radius $\varepsilon/2$, that is $B_{\varepsilon/2}(z)$. We will show that $B_{\varepsilon/2}(z) \subset \mathbf{H}$, proving that $\mathbf{H}$ is opened. Indeed, we can write any $z_0 \in B_{\varepsilon/2}(z)$ as $z_0 = z + z_s$, where $|z_s| < \varepsilon/2$. Thus $\mathrm{Im}(z_0) = \mathrm{Im}(z) + \mathrm{Im}(z_s) > \varepsilon - \varepsilon/2 > 0$.

(b) Observe that f is well-defined as long as $cz + d \neq 0$. Since $(c, d) \neq (0, 0)$ we have two possibilities:

- $c = 0$. In this case $d \neq 0$ and we conclude that f is well-defined.
- $c = 0$. In this case we observe that if $cz + d = 0$ then $z = -\frac{d}{c}$. But $\mathrm{Im}(-d/c) = 0$ so this z does not belong to $\mathbf{H}$, which concludes the proof.

(c) To compute the imaginary part of $f(z)$ we denote $z = u + iv$.

$$\begin{aligned}
\mathrm{Im}\left(\frac{az+b}{cz+d}\right) &= \mathrm{Im}\left(\frac{au+b+iav}{cu+d+icv} \cdot \frac{cu+d-icv}{cu+d-icv}\right) \\
&= \frac{\mathrm{Im}(acu^2 + adu + bcu + bd + acv^2 + i(acuv + adv - acvu - cbv))}{|cz+d|^2} \\
&= \frac{v(ad - cb)}{|cz+d|^2} = \frac{\det(A)v}{|cz+d|^2}.
\end{aligned}$$

Since $v > 0$ we conclude that:

- if $\det(A) > 0$ then f maps $\mathbf{H}$ to $\mathbf{H}$;
- if $\det(A) < 0$ then f maps $\mathbf{H}$ to $\bar{\mathbf{H}}$;
- if $\det(A) = 0$ then $\mathrm{Im}(f(z)) = 0$. In this case we can also conclude that $\mathrm{Re}(f(z))$ is constant. Indeed,

$$\begin{aligned}\mathrm{Re}(f(z)) &= \frac{acu^2 + acv^2 + bd + adu + bcu}{c^2 + 2cdu + d^2 + c^2v^2} = \frac{\frac{a}{c}\left(c^2u^2 + c^2v^2 + \frac{bdc}{a} + 2cdu\right)}{c^2u^2 + 2cdu + d^2 + c^2v^2}\\ &= \frac{a}{c},\end{aligned}$$

where we used that $ad = bc$ in the last two equalities.

(d) We can prove that f is holomorphic by computing the Cauchy-Riemann equations. Another possibility is observing that $g(z) = az + b$ and $h(z) = cz + d$ are holomorphic, since they are polynomials in z. Since $h(z) \neq 0$in $\mathbf{H}$ we can conclude that $\frac{1}{h(z)}$ is also holomorphic and consequently is the product $f = \frac{g}{h}$. The derivative of f is given by:

$$f'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{\det(A)}{(cz + d)^2}.$$

(e) We know that if $\det(A) = 0$ then f is constant. We suppose then that $\det(A) = ad - cb \neq 0$. In this case, observe that f is injective. Indeed, if let $z, w \in \mathbf{H}$ then

$$\begin{aligned}\frac{az + b}{cz + d} = \frac{aw + b}{cw + d} &\Leftrightarrow (az + b)(cw + d) = (aw + b)(cz + d)\\ &\Leftrightarrow aczw + adz + bcw + bd = aczw + bcz + adw + bd\\ &\Leftrightarrow (ad - bc)z = (ad - bc)w \Leftrightarrow w = z.\end{aligned}$$

Observe that in the last implication it was essencial that $ad - bc \neq 0$.
To prove that f is surjective we suppose that $\det(A) > 0$ and we take $w \in \mathbf{H}$. We want to find $z \in \mathbf{H}$ such that

$$\frac{az + b}{cz + d} = w \Rightarrow az + b = czw + wd \Rightarrow z(a - cw) = dw - b.$$

Since $\det(A) \neq 0$, we can't have $(a, c) = (0, 0)$ and thus we can conclude that

$$z = \frac{dw - b}{-cw + a}.$$

To finish the surjectivity proof we just need to check that $z \in \mathbf{H}$, that is $\mathrm{Im}(z) > 0$. We have

$$\mathrm{Im}(z) = \mathrm{Im}(w) \cdot \frac{ad - bc}{|cw - a|^2}.$$

Since $\mathrm{Im}(w) \cdot (ad - bc) > 0$ the result follows. We also deduce that in this case

$$f^{-1}(z) = \frac{dw - b}{-cw + a}.$$

We observe that the function f^{-1} is defined exactly as f with the matrix $\det(\mathrm{A}) \cdot A^{-1}$ in A's place.

When considering that case $\det(A) < 0$ the proof follows anologously.