Exercise sheet 2

Exercise worth bonus points: Exercises 3 and 5

1. Let U and V be open subsets of $\mathbf{C}$. Let $f \in \mathcal{H}(U)$ and $g \in \mathcal{H}(V)$ be holomorphic functions. If $f(U) \subset V$, show that the function from U to $\mathbf{C}$ defined by $F(z) = g(f(z))$ is holomorphic on U and that

$$F'(z) = f'(z)g'(f(z))$$

for all $z \in U$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a $\mathbb{R}$-linear map induced by the matrix

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

That is, for $z = x+iy$, we have $f(x+iy) = \alpha x + \beta y + i(\gamma x + \delta y)$. Find two complex numbers $a, b \in \mathbb{C}$, so that for every $z \in \mathbb{C}$ it holds: $f(z) = az + b\bar{z}$.

3. Let U be open in $\mathbf{C}$ and let $f \in \mathcal{H}(U)$ be a holomorphic function on U. Write $f = u + iv$ with u, v real-valued. If u and v are of class C^2 on U, show that

$$\Delta(u) = \Delta(v) = 0,$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator.

4. Let $u\colon \mathbf{C} \to \mathbf{R}$ be a real-valued function on $\mathbf{C}$.

   (a) Show that there is at most one holomorphic function $f\colon \mathbf{C} \to \mathbf{C}$ such that $\mathrm{Re}(f) = u$ and $\mathrm{Im}(f(0)) = 0$.

   (b) Give an example of a C^∞ function u such that there is no f as in the previous item.

5. (a) Let $n \in \mathbf{Z}$. Compute the integrals

$$\int_\gamma z^n dz$$

where γ is a circle centered at 0 with positive radius, taken counterclockwise.

(b) Let a and b be complex numbers with $|a| < |b|$. Compute

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz$$

where γ is a circle of radius $r \in (|a|, |b|)$ around 0, with counterclockwise orientation.