1. $f \in \mathcal{H}(U)$, $g \in \mathcal{H}(V)$ holomorphic

$f(U) \subset V \Rightarrow F(z) = g(f(z))$ is holomorphic in $U$.

Take $z_0 \in U$. We want to compute the limit:

$$\lim_{h \to 0} \frac{g(f(z_0+h)) - g(f(z_0))}{h} =$$

Observe that if $\forall\, \delta > 0$ $\exists$ $h \in B_\delta(0)$ s.t.

$f(z_0+h) - f(z_0) = 0 \Rightarrow$ since we are assuming that $f$ has derivative in $z_0$ then $f'(z_0) = 0$. In this case it is also clear that $F'(z_0) = 0$ so the formula holds.

Now suppose that $\exists$ $\delta > 0$ s.t. $f(z_0+h) \neq f(z_0)$ $\forall\, h \in B_\delta(0)$.

Then, we have

$$\lim_{h \to 0} \frac{g(f(z_0+h)) - g(f(z_0))}{h} \cdot \frac{f(z_0+h) - f(z_0)}{f(z_0+h) - f(z_0)} \cdot$$

$$= \lim_{h \to 0} \frac{g(f(z_0+h)) - g(f(z_0))}{f(z_0+h) - f(z_0)} \cdot \frac{f(z_0+h) - f(z_0)}{h}$$

$$\boxed{\begin{array}{l} k = f(z_0+h) - f(z_0) \\ h \to 0 \Rightarrow k \to 0 \end{array}}$$

$$\lim_{k \to 0} \frac{g(f(z_0)+k) - g(f(z_0))}{k} \qquad \longrightarrow \quad f'(z_0)$$

$$= g'(f(z_0))$$

2. Let $z = x+iy \Rightarrow$

$$f(x+iy) = \alpha x + \beta y + i(\gamma x + \delta y)$$
$$= (\alpha + i\gamma)x + iy(\delta - \beta i)$$
$$= a(x+iy) + b(x-iy)$$
$$= (a+b)x + i(a-b)y$$

Then $\alpha + i\gamma = a+b$, that is
$\delta - \beta i = a - b$

$$\boxed{a = \frac{\alpha + \delta + i(\gamma - \beta)}{2} \qquad b = \frac{\alpha - \delta + i(\gamma + \beta)}{2}}$$

3. $f \in \mathcal{H}(U)$, $f = u + iv$

Using Cauchy-Riemann eq. we obtain:

$$\frac{\partial^2}{\partial x} u(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} v(x,y)\right)$$

the function is $C^2$ $\rightarrow$ $= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} v(x,y)\right) = \frac{\partial}{\partial y}\left(-\frac{\partial}{\partial y} u(x,y)\right)$

$= -\frac{\partial^2}{\partial y^2} u(x,y)$, which is what we wanted to show.

For the function $v$ we do the analogous computation.

4 $u : \mathbb{C} \longrightarrow \mathbb{R}$

a) at most 1 $f : \mathbb{C} \to \mathbb{C}$ s.t. $Re(z) = u$ $Im(f(0)) = 0$.

Suppose there are two different functions $f_i : \mathbb{C} \to \mathbb{C}$, $i = 1, 2$ s.t.

$f_1 = u + iv_1$ holomorphic in $\mathbb{C}$ w/
$f_2 = u + iv_2$ $v_1(0) = v_2(0) = 0$.

Then $f = f_1 - f_2$ is also holomorphic in $\mathbb{C}$ and satisfies Cauchy-Riemann equations.

$$f(x,y) = i(v_1(x,y) - v_2(x,y)) =: i\, w(x,y)$$

and

$$\frac{\partial}{\partial x} w(x,y) = 0 \quad \text{and} \quad \frac{\partial w}{\partial y}(x,y) = 0$$

$\Rightarrow$ $f'(z) = 0$ and we conclude that $f$ is constant. Since $v_1(0) = v_2(0) = 0$ we conclude that $f_1 \equiv f_2$, which is a contradiction.

b) Let $u(x,y) = x^2$ and suppose $\exists$ $f$ holomorphic s.t

$$f = u + iv.$$

From Cauchy-Riemann we get

$$\frac{\partial v}{\partial y} = 2x \quad \text{and} \quad \frac{\partial v}{\partial x} = 0.$$ Since $v$ is $C^1$

we get

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}(2x) = 2$$

$$\shortparallel$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = 0$$ which is a contradiction.

5a) $$\int_\gamma z^n \, dz, \quad \text{w/} \quad \gamma : [0, 2\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto R \cdot e^{it}, \quad R > 0$$

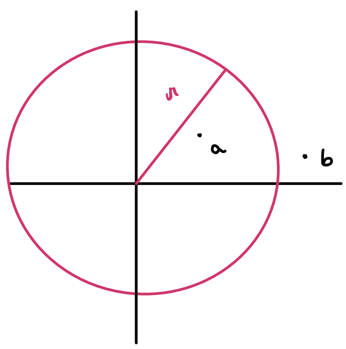
$$= \int_0^{2\pi} R^n \cdot e^{itn} \cdot i \cdot R \cdot e^{it} \, dt = \int_0^{2\pi} R^{n+1} \cdot i \cdot e^{it(n+1)} \, dt$$

$$= \begin{cases} 0 & \text{if} \quad n \neq -1 \\ 2\pi i & \text{if} \quad n = -1 \end{cases}$$

5b $a, b \in \mathbb{C}$ w/ $|a| < |b|$. Compute

$$I = \int_\gamma \frac{1}{(z-a)(z-b)}\, dz$$ for $\gamma$ a circle of radius

$r \in (|a|, |b|)$




We can write

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-b} - \frac{1}{z-a}\right).$$

Then

$$I = \frac{1}{a-b}\int_\gamma \left(\frac{1}{z-b} - \frac{1}{z-a}\right) dz$$

$$= \frac{1}{a-b}\int_0^{2\pi} i\,\theta \cdot r \cdot e^{i\theta}\left\{\frac{1}{re^{i\theta}-b} - \frac{1}{r\cdot e^{i\theta}-a}\right\} d\theta$$

$$= \frac{ir}{a-b}\int_0^{2\pi}\left\{\frac{\theta e^{i\theta}}{re^{i\theta}-b} - \frac{\theta e^{i\theta}}{re^{i\theta}-a}\right\} d\theta$$

Observe that

$$\boxed{\int_0^{2\pi} \frac{e^{i\theta}}{re^{i\theta}-a}\,d\theta} = \int_0^{2\pi} \frac{\cancel{e^{i\theta}}\,d\theta}{r\cdot\cancel{e^{i\theta}}\left(1-\frac{a}{r}\cdot e^{-i\theta}\right)}$$

$\left|\frac{a}{r}\right|<1$ so we can write

$$= \int_0^{2\pi} \frac{1}{r}\cdot\sum_{j=0}^{\infty} (-1)^n \frac{a^n}{r^n}\cdot e^{-i\theta n}$$

$\left(\neq 0 \iff n=0\right)$

$$\boxed{=\frac{2\pi}{r}} \qquad (1)$$

To compute the second integral we do something similar.

$$\int_\gamma \frac{dz}{z-b} = \int_0^{2\pi} \frac{i\cdot r\cdot e^{i\theta}}{r\cdot e^{i\theta}-b}\,d\theta = i\int_0^{2\pi} \frac{r\cdot e^{i\theta}}{b\left(\frac{r}{b}e^{i\theta}-1\right)}\,d\theta$$

Observe that $|r/b|<1$ so we can use the Taylor expansion of $(1-x)^{-1}$ once more:

$$\int_0^{2\pi} e^{i\theta}\sum_{j=0}^{\infty}\frac{r^j}{b^j}\cdot e^{i\theta j}\,d\theta \qquad (2).$$

$$= \int \sum \frac{r^j}{b^j}\, e^{i\theta(j+1)} = 0 \qquad \forall\ j\geq 0.$$

$0 \quad j=0 \quad b^j$

Thus, we conclude using (1) and (2):

$$\int_\gamma \frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left\{ \int_\gamma \frac{dz}{z-a} - \int_\gamma \frac{dz}{z-b} \right\}$$

$$= \frac{1}{a-b} \cdot \left\{ 2\pi i - 0 \right\} = \frac{2\pi i}{a-b}$$