Exercise sheet 3

Exercise worth bonus points: Exercise 3

1. Let U be a non-empty *connected* open set in $\mathbf{C}$ and let $z_0 \in U$. Define a function $f\colon U \to \{0,1\}$ by
$$f(z) = \begin{cases} 1 & \text{if there exists a (piecewise smooth) curve in } U \text{ from } z_0 \text{ to } z \\ 0 & \text{otherwise.} \end{cases}$$
Intuitively, $f(z) = 1$ means that z can be joined to z_0 by a curve.
   (a) Show that the set of $z \in U$ with $f(z) = 1$ is open.
   (b) Let (z_n) be a sequence in U with $f(z_n) = 1$ for all n, and assume that (z_n) converges to $w \in U$. Show that $f(w) = 1$. (Hint: find a small open disc D around w contained in U and some n_0 such that $z_{n_0} \in D$, then combine curves.)
   (c) Deduce that f is continuous and that $f(z) = 1$ for all $z \in U$.

2. Compute the line integrals $\int_\gamma f(x)dz$ for the following functions f and curves $\gamma : [0,1] \to \mathbb{C}$:
   (a) $f(z) = z^2$ and $\gamma(t) = re^{\pi it}, r > 0$.
   (b) $f(z) = \overline{z}$ and $\gamma(t) = \frac{1}{t+1} + it^2$.
   (c) $f(z) = \sin(\mathrm{Re}(z))$ and $\gamma(t) = t + i\sin(t)$.
   (d) $f(z) = z^{-n}$ and $\gamma(t) = e^{2\pi it}$.

3. (a) Show that the function $f(z) = e^{iz^2}$ is holomorphic on $\mathbf{C}$.
   (b) Let $R > 0$ be a real number. Sketch the curve γ consisting of the segment $[0,R] \subset \mathbf{C}$ followed by the arc $t \mapsto Re^{it}$ for $0 \leqslant t \leqslant \pi/4$, and then the segment from $Re^{i\pi/4}$ to 0.
   (c) Compute $\int_\gamma f(z)dz$.
   (d) Deduce that
$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4},$$
(where the improper Riemann integrals are defined as the limit as $R \to +\infty$ of the integral over $[0,R]$).

4. (a) Explain why the function f defined by $f(x) = \sin(x)/x$ for a real number $x \neq 0$ and $f(0) = 1$ is continuous.

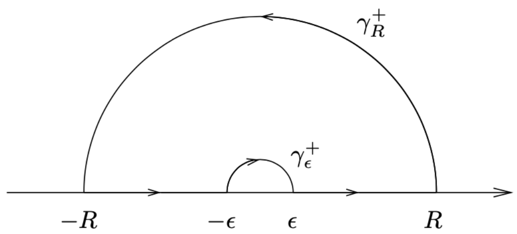
(b) Show that

$$\int_0^\infty \frac{\sin(x)}{x} dx = \lim_{R \to +\infty} \frac{1}{2i} \int_{-R}^{R} \frac{e^{ix} - 1}{x} dx.$$

(c) Let $R > 0$ be large and $\varepsilon > 0$ be small. Explain why

$$\int_\gamma \frac{e^{iz} - 1}{z} dz = 0,$$

where γ is the "indented semicircle" curve described in the picture below.



(d) Deduce the value of

$$\int_0^\infty \frac{\sin(x)}{x} dx.$$