1

a) Set $\Omega = \{ z \in U \mid f(z) = 1 \}$ and let $w \in \Omega$. Since U is open we can take $\varepsilon > 0$ s.t. $B_\varepsilon(w) \subset U$.

For any $w_0 \in B_\varepsilon(w)$ we can consider the curve:

$$\gamma : [0,1] \longrightarrow \mathbb{C}$$
$$t \longmapsto t w_0 + (1-t) w .$$

Observe that $\forall \; t \in [0,1]$

$$|\gamma(t) - w| = | t w_0 - t w | = |t| \cdot |w_0 - w| < \varepsilon \quad \text{so}$$

$$\{ \gamma(t) \, ; \, t \in [0,1] \} \subset B_\varepsilon(w) \subset U .$$

Let $\tilde{\gamma} : [0,1] \longrightarrow \mathbb{C}$ be the piecewise C^1 curve connecting z_0 to w. We define

$$\beta : [0,2] \longrightarrow \mathbb{C}$$

$$\beta(t) = \begin{cases} \tilde{\gamma}(t) & \text{for } t \in [0,1] \\ \gamma(t-1) & \text{for } t \in (1,2] \end{cases} .$$

It's clear that β is piecewise C^1 and connects z_0 to w_0, showing what we wanted.

b) Let $(z_n) \subset U$, $f(z_n) = 1$ and $z_n \to w$ in U.

Since U is open we can take $\varepsilon > 0$ s.t. $B_\varepsilon(w) \subset U$.

$z_n \longrightarrow w$ so there exists $n_0 \in \mathbb{N}$ big enough so that $z_{n_0} \in B_\varepsilon(w)$. To conclude we proceed like in the previous item. We can take a curve joining z_0 and z_{n_0} and another joining z_0 and w. The joint curve concludes the proof.

c) First observe that $\Omega \neq \phi$. Since U is open $\exists\ \varepsilon > 0$ s.t $B_\varepsilon(z_0) \subset U$ and $\forall\ z \in B_\varepsilon(z_0)$, $f(z) = 1$.
If $U = \Omega$ we conclude the result. So suppose by contradiction that this is not true. Thus

$$U = \Omega \cup \Omega_0$$

where $\Omega_0 = \{z : f(z) = 0\} = (U \setminus \Omega) \cap U$ and $\Omega_0 \neq \phi$.

Let $z \in \Omega_0$ and suppose that $\forall\ \varepsilon > 0$ $B_\varepsilon(z) \not\subset \Omega_0$. So, there exists $(z_n) \subset U$ s.t. $z_n \subset \Omega$ and $z_n \xrightarrow{n \to \infty} z$. But this contradicts b) so $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(z) \subset \Omega_0$ and we conclude that Ω_0 is open. Since $\Omega_0 \cap \Omega = \phi$ and Ω is also open we get a contradiction, since U is connected.

2) a) $$\int_\gamma f(z)\,dz = \int_0^1 (r \cdot e^{\pi i t})^2 \cdot \pi \cdot i \cdot r \cdot e^{\pi i t}\,dt$$

$$= \pi r^3 \cdot i \int_0^1 e^{3\pi i t}\,dt = \cancel{\pi} r^3 \cdot \cancel{i} \cdot \left.\frac{e^{3\pi i t}}{3\cancel{\pi}\cancel{i}}\right|_0^1 = \frac{r^3}{3}\left(e^{3\pi i} - 1\right) = \frac{-2r^3}{3}.$$

b) $\int_{\gamma} \bar{z}\, dz = \int_0^1 \left(\frac{1}{t+1} - it^2 \right) \cdot \left(-\frac{1}{(t+1)^2} + 2it \right) dt$

$$= \int_0^1 \left(\frac{-1}{(t+1)^3} + \frac{2it}{t+1} + \frac{it^2}{(t+1)^2} + 2t^3 \right) dt$$

$$= -3/8 + 2i(1 - \log 2) + i\left(3/2 - \log 4\right) + 1/2$$

$$= 1/8 + i\left(7/2 - 4\log 2\right).$$

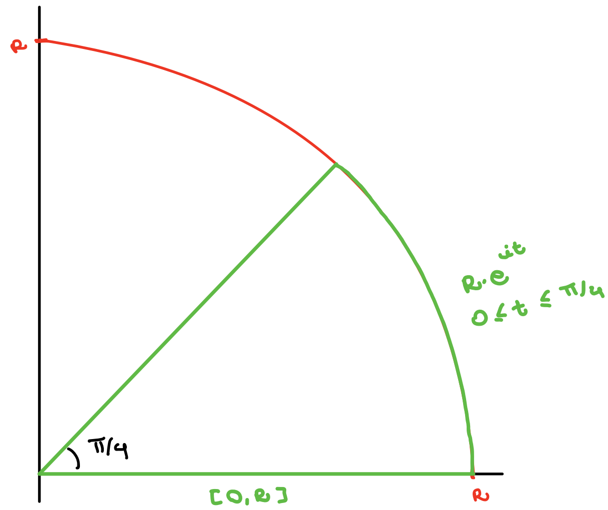
c) $\int_{\gamma} f(z)\, dz = \int_0^1 \sin t \cdot (1 + i\cos t)\, dt = 1 - \cos(t) + i\frac{\sin^2(1)}{2}.$

d) $\int_0^1 2\pi i \cdot \frac{e^{2\pi i t}}{e^{2\pi i m t}} dt = \begin{cases} 2\pi i & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases}$

3 a) $f(z) = e^{iz^2}$ is holomorphic in $\mathbb{C}$:

We know that $f_1(z) = e^{z}$ and $f_2(z) = iz^2$ are holomorphic in $\mathbb{C}$, so $f_1 \circ f_2(z)$ is as well.

b)



c) From Cauchy's thm we get that

$$\int_\gamma f(z)\,dz = 0 .$$

d) Observe that

$$0 = \int_\gamma f(z)\,dz = \int_0^R e^{ix^2}\,dx + \overbrace{\int_0^{\pi/4} e^{i(R\cdot e^{i\theta})^2}\, i\, R\cdot e^{i\theta}\,d\theta}^{\mathrm{I}}$$

$\boxed{\text{Main}}$

$$\underbrace{-\int_0^1 e^{i(1-t)^2 R^2\cdot e^{i\pi/2}} \cdot R\cdot e^{i\pi/4}\,dt}_{\mathrm{II}}$$

We analise I:

$$\left| \int_0^{\pi/4} e^{i(R\cdot e^{i\theta})^2} \, i\, R\cdot e^{i\theta} \, d\theta \right| \leq \left| \int_0^{\pi/4} R\, e^{-R^2 \cdot \sin(2\theta)} \, d\theta \right|$$

Let $\delta > 0$. Then,

$$\int_0^{\pi/4} R\, e^{-R^2 \sin(2\theta)} \, d\theta = \underbrace{\int_0^{\delta} R\cdot e^{-R^2 \sin 2\theta} \, d\theta}_{*} + \underbrace{\int_{\delta}^{\pi/4} R\cdot e^{-R^2 \sin 2\theta} \, d\theta}_{**}$$

$$* \quad \int_0^{\delta} R\cdot e^{-R^2 \sin 2\theta} \, d\theta \leq R\cdot\delta$$

$\left(e^{-R^2 \sin\theta} \leq 1 \text{ because } \sin 2\theta \geq 0 \text{ for } 0 \leq \theta \leq \pi/4 \right)$

$$** \quad \int_{\delta}^{\pi/4} R\cdot e^{-R^2 \sin 2\theta} \, d\theta \leq \int_{\delta}^{\pi/4} R\cdot e^{-R^2 \sin(2\delta)} \, d\theta$$

$$= (\pi/4 - \delta)\, R\cdot e^{-R^2 \sin(2\delta)}$$

If we take R big enough so that $R\sin(2\delta/R) \geq \delta$ then

$$* + ** \leq R\cdot\delta + (\pi/4 - \delta)\cdot R\cdot e^{-R^2 \sin(2\delta)}$$

Picking $\delta = 1/R^2$ we get:

$$\left| \int_0^{\pi/4} R \cdot e^{-R^2 \sin 2\theta} \, d\theta \right| \leq \frac{1}{R} + (\pi/4 - 1/R) \cdot R \cdot e^{-R^2 \sin(2/R)}$$

$$\longrightarrow 0 \quad \text{as} \quad R \to \infty$$

We analyse II:

$$-\int_0^1 e^{i(1-t)^2 R^2 \cdot e^{i\pi/2}} \cdot R \cdot e^{i\pi/4} \, dt = -R \cdot e^{i\pi/4} \int_0^1 e^{-(1-t)^2 R^2} \, dt$$

$$= -\frac{1}{\sqrt{2}} \cdot (1+i) \cdot \int_0^R e^{-x^2} \, dx .$$

Taking $R \to \infty$ in Main we get

$$\lim_{R \to \infty} \int_0^R e^{ix^2} \, dx = -\lim_{R \to \infty} \int_0^{\pi/4} e^{i(R \cdot e^{i\theta})^2} \, i R \cdot e^{i\theta} \, d\theta$$

$$+ \lim_{R \to \infty} \int_0^1 e^{i(1-t)^2 R^2 \cdot e^{i\pi/2}} \cdot R \cdot e^{i\pi/4} \, dt$$

$$\longrightarrow 0$$

$$\Rightarrow \int_0^\infty e^{ix^2} \, dx = \frac{(1+i)}{\sqrt{2}} \lim_{R \to} \int_0^R e^{-x^2} \, dx = \frac{(1+i)}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{2}$$

$$= (1+i) \sqrt{2\pi} / 4$$

Taking the Real and Imaginary parts of the integral we conclude

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

4 a) We compute the limit:

$$(\star)\ \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1 .$$ Since

$\frac{\sin x}{x}$ is continuous in $\mathbb{R}\setminus\{0\}$ and

$(\star)$ holds, we conclude that f is continuous.

b) Observe that

$$\frac{1}{2i}\int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx = \frac{1}{2i}\int_{-R}^{R} \frac{\cos x - 1 + i\sin x}{x}\,dx$$

Taking the limit $R \longrightarrow \infty$ we get

$$\lim_{R\to\infty} \frac{1}{2i}\int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{\cos x - 1}{x}\,dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

Since $\dfrac{\cos x - 1}{x}$ is an odd function we get

$$\int_{-\infty}^{\infty} \frac{\cos x - 1}{x}\, dx = 0$$ and since $\dfrac{\sin x}{x}$ is even

$$-\frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \int_{0}^{\infty} \frac{\sin x}{x}\, dx .$$

c) We note that $g(z) = \dfrac{e^{iz} - 1}{z}$ is

holomorphic in $\mathbb{C} \setminus \{0\}$. Since γ and the region it encloses are contained in $\mathbb{C} \setminus \{0\}$, by Cauchy's therm, we conclude that

$$\int_{\gamma} g(z)\, dz = 0 .$$

d) From the previous item we have:

$$\int_{\varepsilon}^{R} \frac{e^{ix} - 1}{x}\, dx + \int_{-R}^{-\varepsilon} \frac{e^{ix} - 1}{x}\, dx = \underbrace{-\int_{\gamma_R^+} \frac{e^{iz} - 1}{z}\, dz}_{I}$$

$$+\underbrace{\int_{\gamma_\varepsilon^+} \frac{e^{iz}-1}{z}\,dz}_{\mathrm{II}}$$

We analyse I: paramatrizing γ_R as:

$$\gamma_R : [0,\pi] \longrightarrow \mathbb{C}$$
$$t \longmapsto R\cdot e^{ti}$$

$$\int_{\gamma_R^+} \frac{e^{iz}-1}{z}\,dz = \int_0^\pi \frac{e^{iR\cdot e^{it}}-1}{\cancel{R}\cdot\cancel{e^{ti}}} \cdot i\cdot\cancel{R}\cdot\cancel{e^{ti}}\;dt$$

$$= i\int_0^\pi \left\{ e^{iR\cos t - R\sin t} - 1 \right\} dt.$$

$$= -\pi i + i\int_0^\pi e^{iR\cos t - R\sin t}\,dt$$

Observe that

$$\left| \int_0^\pi e^{iR\cos t - R\sin t}\,dt \right| \leq \int_0^\pi e^{-R\sin t}\,dt.$$

For $t \in [0, \pi]$, $\sin t \geq 0$, thus $e^{-R\sin t} \leq 1$

We can use the Dominated Convergence Term and conclude that

$$\lim_{R \to \infty} \int_0^{\pi} e^{-R\sin t} \, dt = 0 \quad \text{or}$$

we take $\delta > 0$ and split the integral:

$$\int_0^{\delta} e^{-R\sin t} \, dt + \int_{\delta}^{\pi} e^{-R\sin t} \, dt \leq \delta + e^{-R\sin(\delta)} \cdot (\pi - \delta).$$

Taking $\delta = 1/\log R$ we get the result.

Indeed:

$$\lim_{R \to \infty} R \cdot \sin\left(\frac{1}{\log R}\right) = \lim_{R \to \infty} \frac{\sin(1/\log R)}{1/R}$$

$$= \lim_{R \to \infty} \frac{\cos(1/\log R) \cdot 1/(\log R)^2 \cdot 1/R}{1/R^2} =$$

$$\lim_{R \to \infty} \frac{R \cdot \cos(1/\log R)}{(\log R)^2} = \infty.$$

For II we get:

$$\int_{\gamma_\varepsilon^+} \frac{e^{iz}-1}{z}\,dz = \int_0^\pi \frac{e^{i\varepsilon\cdot e^{it}}-1}{\varepsilon\cdot e^{it}}\cdot \varepsilon\cdot i\cdot e^{it}\,dt$$

$$= i\int_0^\pi \left(e^{i\varepsilon\cos t}\cdot e^{-\varepsilon\sin t}-1\right)dt$$

$$= -\pi i + i\int_0^\pi e^{i\varepsilon\cos t}\cdot e^{-\varepsilon\sin t}\,dt.$$

Taking the limit when $\varepsilon \longrightarrow 0$ we get

$$\lim_{\varepsilon\to 0}\int_0^\pi e^{i\varepsilon\cos t-\varepsilon\sin t}\,dt = \int_0^\pi \lim_{\varepsilon\to 0} e^{i\varepsilon\cos t}\cdot e^{-\varepsilon\sin t}\,dt = \pi$$

Dominated Convergence Thm, because $|e^{i\varepsilon\cos t}\cdot e^{-\varepsilon\sin t}| \le 1$, for $t\in[0,\pi]$.

Putting everything together we get the result.