Exercise sheet 4

Exercise worth bonus points: Exercises 2 and 4

1. (Comparison between vector calculus from Analysis II and Complex Analysis) Let $U \subset \mathbb{R}^2$ be an open set, $V = (V_1, V_2) : U \to \mathbb{R}^2$ a C^1 vector field on U, $\gamma : [a, b] \to U$ a C^1 curve and $h : U \to \mathbb{R}$ a C^1 function.

   We recall the following definitions from Analysis II:

   $$\operatorname{div} V = \partial_x V_1 + \partial_y V_2 \qquad \operatorname{rot} V = \partial_x V_2 - \partial_y V_1$$

   $$\operatorname{grad} h = (\partial_x h, \partial_y h) \qquad \int_\gamma V \cdot d\mathbf{s} = \int_a^b V(\gamma(t)) \cdot \dot{\gamma}(t) dt$$

   Now let $f = u + iv : U \to \mathbb{C}$ be a complex valued continuous function. We define the following vector fields:

   $$V_f := (u, -v) \qquad W_f := (v, u).$$

   One observes that we can obtain W_f from V_f by a $\pi/2$-rotation.

   Prove the following assertions:

   (a) Assume that f is differentiable and of class C^1 in the sense of Analysis II. Then we have

   $$\begin{aligned} f \text{ holomorphic} &\Leftrightarrow \operatorname{div} V_f = \operatorname{rot} V_f = 0 \Leftrightarrow \operatorname{div} W_f = \operatorname{rot} W_f = 0 \\ &\Leftrightarrow \operatorname{div} V_f = \operatorname{div} W_f = 0 \Leftrightarrow \operatorname{rot} V_f = \operatorname{rot} W_f = 0. \end{aligned}$$

   (b) The complex line integral can be expressed as the following real line integral:

   $$\int_\gamma f(z) dz = \int_\gamma V_f \cdot d\mathbf{s} + i \int_\gamma W_f \cdot d\mathbf{s}.$$

   (c) If f is holomorphic on U and $g = f'$, then

   $$V_g = \operatorname{grad} u \qquad W_g = \operatorname{grad} v.$$

2. Let $f \in \mathcal{H}(\mathbf{C})$ be a holomorphic function on $\mathbf{C}$. Suppose that there exists an integer $d \geqslant 1$ and a real number $C \geqslant 0$ such that

   $$|f(z)| \leqslant C(1 + |z|)^d$$

   for all $z \in \mathbf{C}$. Prove that f is a polynonial of degree at most d.

3. Let U be a connected open set in $\mathbf{C}$ and let f be a holomorphic function on U such that $f' = 0$.

   (a) Show that f is constant.

   (b) Explain why it is important to assume that U is connected.

4. Let U be the open set of $z \in \mathbf{C}$ such that $z \neq 2ik\pi$ for some *non-zero* $k \in \mathbf{Z}$. Let $f(z) = z/(e^z - 1)$ for $z \in U$ non-zero and $f(0) = 1$.

   (a) Show that $f \in \mathcal{H}(U)$. (Hint: to show that f is holomorphic close to 0, express $e^z - 1 = zg(z)$ where g is holomorphic and non-zero at 0).
   We denote by
$$\sum_{n=0}^{+\infty} \frac{b_n}{n!} z^n$$
   the Taylor series for f at 0.

   (b) Compute b_0, b_1, b_2, b_3.

   (c) Prove that the radius of convergence of the Taylor series is 2π.

   (d) For any r with $r > 2\pi$, deduce that there are infinitely many integers n such that $|b_n| \geqslant n!/r^n$.

5. Compute the radius of convergence of the Taylor series of the function
$$f(z) = \frac{1}{z^3 + 2}$$
   at $z_0 = 0$ and at $z_0 = 1$.

6. Let U be a non-empty connected open set and let f, g be holomorphic functions on U. If $fg = 0$, then either $f = 0$ or $g = 0$.