Exercise sheet 4 - solutions

1 a) Observe that $u_x = v_y$ $u_y = -v_x$ holds if and only if

$$\operatorname{div} V_f = u_x - v_y = 0 \quad \text{and} \quad \operatorname{rot}(V_f) = -v_x - u_y = 0$$

if and only if

$$\operatorname{div} W_f = v_x + u_y = 0 \quad \text{and} \quad \operatorname{rot}(W_f) = u_x - v_y = 0$$

b) Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \big(u(\gamma(t)) + i v(\gamma(t)) \big) \gamma'(t)\, dt$$

$$= \int_a^b [u(\gamma(t)) \cdot \gamma_1'(t) - v(\gamma(t)) \cdot \gamma_2'(t)]\, dt +$$

$$i \int_a^b [v(\gamma(t)) \cdot \gamma_1'(t) + u(\gamma(t)) \cdot \gamma_2'(t)]\, dt$$

$$= \int_\gamma V_f \cdot ds + i \int_\gamma W_f \cdot ds$$

c) Let $g = f'$. Then $g(x,y) = u_x(x,y) - i u_y(x,y) = (u_x, -u_y) =$ grad u. It also holds that

$$g(x,y) = v_y(x,y) + i v_x(x,y) = (v_y, u_x) = \operatorname{grad} v.$$

2) $f \in \mathcal{H}(\mathbb{C})$. $\exists\ C > 0$ s.t. $|f(z)| \leq C(1+|z|)^d$ $\forall z \in \mathbb{C}$

$\Rightarrow$ f is a pol. of degree at most d.

Since $f \in \mathcal{H}(\mathbb{C})$, it has a Taylor expansion at z_0, $\forall z_0 \in \mathbb{C}$, that converges in the whole complex plane. Let's compute its Taylor expansion around 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for} \quad a_n = \frac{f^{(n)}(0)}{n!}.$$

Let $n > d$ and C_R be a circle of radius R around 0. Then, Cauchy inequalities imply that:

$$|f^{(n)}(0)| \leq \frac{n!}{R^n} \cdot \sup_{z \in C_R} |f(z)| \underset{\text{hypothesis}}{\leq} \frac{n! \cdot C \cdot (1+R)^d}{R^n}$$

Since $f \in \mathcal{H}(\mathbb{C})$ we can let $R \to \infty$ and we get

$$f^{(n)}(0) = 0 \quad \forall\ n > d, \text{ thus}$$

$$f(z) = \sum_{n=0}^{d} a_n z^n, \text{ as we wanted to show.}$$

3) U connected open in $\mathbb{C}$, f holomorphic on U, $f' = 0$.

a) f is constant.

Let $z_0 \in U$ and $r > 0$ s.t. $B_r(z_0) \subset U$. We know that for $z \in B_r(z_0)$ it holds:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n .$$

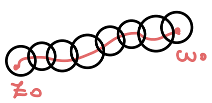
Since $f' \equiv 0$ in $B_r(z_0)$ it holds that $f^{(n)}(z)\Big|_{B_r(z_0)} \equiv 0 \quad \forall\, n \geq 1$.

Thus $f(z) \equiv f(z_0)$ in $B_r(z_0)$.

Now take $w_0 \neq z_0 \in U$. Since U is connected there is $\gamma : [0,1] \longrightarrow U$ continuous, $\gamma(0) = w_0$, $\gamma(1) = z_0$.

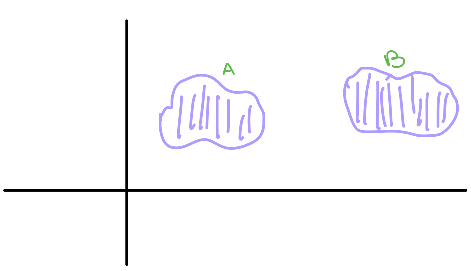
Consider the open covering $\{ B_{r(t)}(\gamma(t)) \}_{t \in [0,1]}$, for $r(t)$ s.t. $B_{r(t)}(\gamma(t)) \subset U$, of $\gamma(U) \subset U$. Since $\gamma(U)$ is compact, $\exists$ finite subcover, $\{ B_{r(t_n)}(\gamma(t_n)) \}_{n=1}^{N}$. From what we showed, it holds that $f(\gamma(t_n)) = f(z) \ \forall$ $z \in B_{r(t_n)}(\gamma(t_n))$. But since the balls intersect we get

$$f\left(B_{r(t_n)}(\gamma(t_n))\right) = f\left(B_{r(t_m)}(\gamma(t_m))\right)$$



$\Rightarrow f(w_0) = f(z_0)$ and we conclude that f is constant in U.

b) Otherwise we could have:



$U = A \cup B$ w/

$f(z) = 1 \quad \forall\, z \in A$

$f(z) = 0 \quad \forall\, z \in B$

$f' \equiv 0$ but f is not constant.

(4) a) $f(z) = \dfrac{z}{e^z - 1} \in \mathcal{H}(U)$ for $U = \mathbb{C} \setminus \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \{2ik\pi\}$

Observe that $e^z = 1 \iff z = 2\pi i k$ for $k \in \mathbb{Z}$. Thus, to show that $f \in \mathcal{H}(U)$, we just need to show that f is holomorphic in $B_r(0)$ for a $0 < r < 2\pi$.

$$e^z - 1 = \sum_{m=1}^{\infty} \frac{z^m}{m!} = z \cdot \underbrace{\sum_{m=1}^{\infty} \frac{z^{m-1}}{m!}}_{g(z)} .$$

$g(z)$ is holomorphic in $\mathbb{C}$ and $g(0) = 1$.

Thus,

$f(z) = \dfrac{z}{z \cdot g(z)} = \dfrac{1}{g(z)}$. Since $g(0) = 1$, $\exists$ $0 < r < 2\pi$ s.t. $g(z) \neq 0$ $\forall$ $z \in B_r(0)$ and we can conclude that f is holomorphic in $B_r(0)$.

b) $f(z) = \displaystyle\sum_{m=0}^{\infty} \frac{b_m}{m!} z^m$, $b_m = f^{(m)}(0)$.

$$f(z) = \frac{z}{e^z - 1} \Rightarrow (e^z - 1)\, f(z) = z$$

$$\Rightarrow \left(\sum_{m=1}^{\infty} \frac{z^m}{m!} \right) \cdot \left(\sum_{m=0}^{\infty} \frac{b_m}{m!} \cdot z^m \right) = z$$

$$\sum_{k=0}^{\infty} c_k \cdot z^k, \text{ where } c_k = \sum_{l+p=k} \frac{1}{l!} \frac{b_p}{p!} .$$

$1 = C_1 = b_0$

$0 = C_2 = \frac{b_0}{2} + b_1 \Rightarrow b_1 = \frac{-1}{2}$

$0 = C_3 = \frac{b_2}{2!} + \frac{b_1}{2!} + \frac{b_0}{3!} \Rightarrow b_2 = \frac{1}{6}$

$0 = C_4 = \frac{b_0}{4!} + \frac{b_1}{3!} + \frac{b_2}{2!2!} + \frac{b_3}{3!} \Rightarrow b_3 = 0$

c) Observe that f is holomorphic in $B_{2\pi}(0)$ and it is not holomorphic in $\pm 2\pi$, thus the radius of convergence is 2π.

d) Fix $r > 2\pi$ and suppose that $|b_m| \leq m!/r^m$ $\forall$ $m \geq m_0$. For $|z| < r$ we then have

$$\left| \sum_{m=0}^{\infty} \frac{b_m}{m!} z^m \right| = \underbrace{\left| \sum_{m=0}^{m_0} \frac{b_m \cdot z^m}{m!} \right|}_{< +\infty} + \underbrace{\sum_{m=m_0+1}^{\infty} \frac{|b_m|}{m!} |z|^m}_{\leq \sum_{m=m_0+1}^{\infty} \frac{|z|^m}{r^m} < +\infty}$$

because $\frac{|z|}{r} < 1$.

and we would conclude that f is holomorphic in $B_r(0)$, which is a contradiction.

5 We compute the roots of z^3+2.

$$z^3+2=0 \Rightarrow z^3=-2 \Rightarrow z_k = 2^{1/3}\cdot e^{i\pi/3\cdot k}, \quad k=0,1,2$$

$$|z_k| = 2^{1/3}$$

$\Rightarrow$ the radius of convergence for $z=0$ is $2^{1/3}$.

For $z=1$ we have:

$$|z-z_0| = |1+2^{1/3}|$$

$$|z-z_2| = |z-z_3| = \left[\left(1-1/2^{2/3}\right)^2 + 3/2^{4/3} \right]^{1/2}$$

$$= \left(1 - \frac{2}{2^{2/3}} + \frac{1}{2^{4/3}} + \frac{3}{2^{4/3}} \right)^{1/2}$$

$$= \left(\frac{2^{4/3} - 2^{5/3} + 4}{2^{4/3}} \right)^{1/2} = \left(1 - 2^{1/3} + 2^{2/3}\right)^{1/2}$$

$$\Rightarrow r = |z-z_2|$$

6 Suppose $f \not\equiv 0$ and $g \not\equiv 0$. Call

$Z_f = \{z \in U : f(z) = 0\}$

$Z_g = \{z \in U : g(z) = 0\}$ have no limit point.

$Z_{f\cdot g} = Z_f \cup Z_g$.

By contradiction, suppose $\exists \{z_k\} \subset Z_{fg}$ w/ $z_k \xrightarrow{k\to\infty} z \in Z_{fg}$. So, without loss of generality we can take subsequence $\{z_{m_k}\} \subset Z_f$, that is

$f(z_{m_k}) = 0 \;\; \forall \, m_k$ and we know that $z \in U$, so by continuity $f(z)=0$ which is a

Contradiction, because $f \not\equiv 0$.